ENERGY DECAY FOR THE DAMPED WAVE EQUATION UNDER A PRESSURE CONDITION

EMMANUEL SCHENCK

ABSTRACT. We establish the presence of a spectral gap near the real axis for the damped wave equation on a manifold with negative curvature. This results holds under a dynamical condition expressed by the negativity of a topological pressure with respect to the geodesic flow. As an application, we show an exponential decay of the energy for all initial data sufficiently regular. This decay is governed by the imaginary part of a finite number of eigenvalues close to the real axis.

1. INTRODUCTION

One of the standard questions in geometric control theory concerns the so-called stabilization problem: given a dissipative wave equation on a manifold, one is interested in the behaviour of the solutions and their energies for long times. The answers that can be given to this problem are closely related to the underlying manifold and the geometry of the control (or damping) region.

In this paper, we shall study these questions in the particular case of the damped wave equation on a compact Riemannian manifold (M, g) with negative curvature and dimension $d \geq 2$. For simplicity, we will assume that M has no boundary. If $a \in C^\infty(M)$ is a real valued function on M, this equation reads

$$(\partial_t^2 - \Delta_g + 2a(x)\partial_t)u = 0\,, \quad (t,x) \in \mathbb{R} \times M\,, \tag{1.1}$$

with initial conditions

$$\begin{aligned} u(0,x) &= \omega_0(x) \in H^1 \\ \mathrm{i}\,\partial_t u(0,x) &= \omega_1(x) \in H^0. \end{aligned}$$

Here $H^s \equiv H^s(M)$ are the usual Sobolev spaces on M. The Laplace-Beltrami operator $\Delta_g \equiv \Delta$ is expressed in local coordinates by

$$\Delta_g = \frac{1}{\sqrt{\bar{g}}}\partial_i(g^{ij}\sqrt{\bar{g}}\partial_j)\,, \quad \bar{g} = \det g. \tag{1.2}$$

We will also denote by $d\mathrm{vol} = \sqrt{\bar{g}}dx$ the natural Riemannian density, and $\langle u, v\rangle = \int_M u\bar{v}d\mathrm{vol}$ the associated scalar product.

In all the following, we will consider only the case where the waves are *damped*, wich corresponds to take $a \geq 0$ with a non identically 0. We can reformulate the above problem into an equivalent one by considering the unbounded operator

$$\mathcal{B} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta_g & -2\,\mathrm{i}\,a \end{pmatrix} : H^1 \times H^0 \to H^1 \times H^0$$

with domain $D(\mathcal{B}) = H^2 \times H^1$, and the following evolution equation :

$$(\partial_t + \mathrm{i}\,\mathcal{B})\mathbf{u} = 0\,, \quad \mathbf{u} = (u_0, u_1) \in H^1 \times H^0. \tag{1.3}$$

From the Hille-Yosida theorem, one can show that $\mathcal{B}$ generates a uniformly bounded, strongly continuous semigroup $\mathrm{e}^{-\mathrm{i}\, t\mathcal{B}}$ for $t \geq 0$, mapping any $(u_0, u_1) \in H^1 \times H^0$ to a solution $(u(t,x), \mathrm{i}\, \partial_t u(t,x))$ of (1.3). Since $\mathcal{B}$ has compact resolvent, its spectrum $\mathrm{Spec}\, \mathcal{B}$ consist in a discrete sequence of eigenvalues $\{\tau_n\}_{n\in\mathbb{N}}$. The eigenspaces E_n corresponding to the eigenvalues τ_n are all finite dimensional, and the sum $\bigoplus_n E_n$ is dense in $H^1 \times H^0$, see [GoKr]. If $\tau \in \mathrm{Spec}\, \mathcal{B}$, there is $v \in H^1$ such that

$$u(t,x) = \mathrm{e}^{-\mathrm{i}\, t\tau}\, v(x)\,, \tag{1.4}$$

and the function u then satisfies

$$P(\tau)u = 0\,, \quad \text{where} \quad P(\tau) = -\Delta - \tau^2 - 2\,\mathrm{i}\, a\tau. \tag{1.5}$$

From (1.5), it can be shown that the spectrum is symmetric with respect to the imaginary axis, and satisfies

$$-2\|a\|_\infty \leq \mathrm{Im}\, \tau_n \leq 0$$

while $|\,\mathrm{Re}\, \tau_n| \to \infty$ as $n \to \infty$. Furthermore, if $\mathrm{Re}\, \tau \neq 0$, we have $\mathrm{Im}\, \tau \in [-\|a\|_\infty, 0]$, and the only real eigenvalue is $\tau = 0$, associated to the constant solutions of (1.1).

The question of an asymptotic density of modes has been adressed by Markus and Matsaev in [MaMa], where they proved the following Weyl-type law, also found later independently by Sjöstrand in [Sjö] :

$$\mathrm{Card}\{n : 0 \leq \mathrm{Re}\, \tau_n \leq \lambda\} = \left(\frac{\lambda}{2\pi}\right)^d \int_{p^{-1}([0,1])} dx d\xi + \mathcal{O}(\lambda^{d-1})\,.$$

Here $p = g_x(\xi, \xi)^2$ is the principal symbol of $-\Delta_g$ and $dxd\xi$ denotes the Liouville measure on T^*M coming from its symplectic structure. Under the asumption of ergodicity for the geodesic flow with respect to the Liouville measure, Sjöstrand also showed that most of the eigenvalues concentrate on a line in the high-frequency limit. More precisely, he proved that given any $\varepsilon > 0$,

$$\mathrm{Card}\{n : \tau_n \in [\lambda, \lambda+1] + \mathrm{i}(\mathbb{R} \setminus [-\bar{a} - \varepsilon, -\bar{a} + \varepsilon])\} = o(\lambda^{d-1})\,. \tag{1.6}$$

The real number $\bar{a}$ is the ergodic mean of a on the unit cotangent bundle $S^*M = \{(x,\xi) \in T^*M, g_x(\xi,\xi) = 1\}$. It is given by

$$\bar{a} = \lim_{T\to\infty} T^{-1} \int_0^T a \circ \Phi^t dt, \text{ well defined } dxd\xi - \text{almost everywhere on } S^*M\,.$$

Hence the eigenvalues close to the real axis, say with imaginary parts in $[\alpha, 0]$, $0 > \alpha > -\bar{a}$ can be considered as "exceptional". The first result we will present in this paper show that a spectral gap of finite width can actually exist below the real axis under some dynamical hypotheses, see Theorem 1 below.

The second object studied in this work is the energy of the waves. From now on, we call $\mathcal{H} = H^0 \times H^1$ the space of Cauchy data. Let u be a solution of (1.1) with initial data $\boldsymbol{\omega} \in \mathcal{H}$. The energy of u is defined by

$$E(u,t) = \frac{1}{2}(\|\partial_t u\|^2_{L^2} + \|\nabla u\|^2_{L^2})\,.$$

As a well known fact, E is decreasing in time, and $E(u,t) \stackrel{t\to\infty}{\longrightarrow} 0$. It is then natural to ask if a particular rate of decay of the energy can be identified. Let $s > 0$ be a positive number, and define the Hibert space

$$\mathcal{H}^s = H^{1+s} \times H^s \subset \mathcal{H}\,.$$

Generalizing slightly a definition of Lebeau, we introduce the best exponential rate of decay with respect to $\|\cdot\|_{\mathcal{H}^s}$ as

$$\rho(s) = \sup\{\beta \in \mathbb{R}_+ : \exists C > 0 \text{ such that } \forall \boldsymbol{\omega} \in \mathcal{H}^s,\ E(u,t) \leq C\,\mathrm{e}^{-\beta t}\,\|\boldsymbol{\omega}\|_{\mathcal{H}^s}\} \tag{1.7}$$

where the solutions u of (1.1) have been identified with the Cauchy data $\boldsymbol{\omega} \in \mathcal{H}^s$. It is shown in [Leb] that

$$\rho(0) = 2\min(G, C(\infty)),$$

where $G = \inf\{-\operatorname{Im}\tau; \tau \in \operatorname{Spec}\mathcal{B} \setminus \{0\}\}$ is the spectral gap, and

$$C(\infty) = \lim_{t\to\infty} \inf_{\rho \in T^*M} \frac{1}{t}\int_0^t \pi^* a(\Phi^s \rho) ds \geq 0.$$

Here $\Phi^t : T^*M \to T^*M$ is the geodesic flow, and $\pi : T^*M \to M$ is the canonical projection along the fibers. It follows that the presence of a spectral gap below the real axis is of significative importance in the study of the energy decay. However, an explicit example is given in [Leb], where $G > 0$ while $C(\infty) = 0$, and then $\rho(0) = 0$. This particular situation is due to the failure of the geometrical control, namely, the existence of orbits of the geodesic flow not meeting $\operatorname{supp} a$ (which implies $C(\infty) = 0$). Hence, the spectrum of $\mathcal{B}$ may not always control the energy decay, and some dynamical assumptions on the geodesic flow are required if we want to solve positively the stabilization problem. In the case where geometric control holds [RaTa], it has been shown in various settings that $\rho(0) > 0$, see for instance [BLR, Leb, Hit]. In [Chr], a particular situation is analyzed where the geometric control does not hold near a closed hyperbolic orbit of the geodesic flow: in this case, there is a sub-exponential decay of the energy with respect to $\|\cdot\|_{\mathcal{H}^\varepsilon}$ for some $\varepsilon > 0$.

Dynamical assumptions. In this paper, we first assume (M, g) has strictly negative sectional curvatures. This implies that the geodesic flow has the Anosov property on every energy layer, see Section 2.1 below. Without loss of generality, we suppose that the injectivity radius satisfies $r \geq 2$. Then, we drop the geometric control assumption, and replace it with a dynamical hypothese involving the topological pressure of the geodesic flow on S^*M, which we define now. For ervery $\varepsilon > 0$ and $T > 0$, a set $\mathcal{S} \subset S^*M$ is $(\varepsilon, T)-$separated if $\rho, \theta \in \mathcal{S}$ implies that $d(\Phi^t\rho, \Phi^t\theta) > \varepsilon$ for some $t \in [0, T]$, where d is the distance induced from the adapted metric on T^*M. For f continuous on S^*M, set

$$Z(f, T, \varepsilon) = \sup_{\mathcal{S}} \left\{ \sum_{\rho \in \mathcal{S}} \exp \sum_{k=0}^{T-1} f \circ \Phi^k(x) \right\}.$$

The topological pressure $\Pr(f)$ of the function f with respect to the geodesic flow is defined by

$$\Pr(f) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log Z(f, T, \varepsilon).$$

The pressure $\Pr(f)$ contains useful information on the Birkhoff averages of f and the complexity of the geodesic flow, see for instance [Wal] for a general introduction and further properties. The particular function we will deal with is given by

$$a^u : \rho \in S^*M \mapsto a^u(\rho) = -\int_0^1 \pi^* a \circ \Phi^s(\rho)\, ds + \frac{1}{2}\log J^u(\rho) \in \mathbb{R} \tag{1.8}$$

where $J^u(\rho)$ is the unstable Jacobian at ρ for time 1, see Section 2.1. In this paper, we will always assume that

$$\Pr(a^u) < 0. \tag{1.9}$$

Main results. Under the condition $\Pr(a^u) < 0$, we will see that a spectral gap of finite width exists below the real axis. As a consequence, there is an exponential decay of the energy of the waves with respect to $\| \cdot \|_{\mathcal{H}^\kappa}$ for any $\kappa \geq d/2$, and if $G < |\Pr(a^u)|$, we have $\rho(\kappa) = 2G$. We begin by stating the result concerning the spectral gap.

Theorem 1. (Spectral gap) *Suppose that the topological pressure of a^u with respect to the geodesic flow on S^*M satisfies $\Pr(a^u) < 0$, and let $\varepsilon > 0$ be such that*

$$\Pr(a^u) + \varepsilon < 0.$$

Then, there exisits $e_0(\varepsilon) > 0$ such that for any $\tau \in \operatorname{Spec} \mathcal{B}$ with $|\operatorname{Re} \tau| \geq e_0(\varepsilon)$, we have

$$\operatorname{Im} \tau \leq \Pr(a^u) + \varepsilon.$$

The presence of a spectral gap of finite width below the real axis is not obvious a priori if geometric control does not hold, since there may be a possibility for $|\operatorname{Im} \tau_n|$ to become arbitrary small as $n \to \infty$: see for instance [Hit], Theorem 1.3. However, this accumulation on the real axis can not occur faster than a fixed exponential rate, as it was shown in [Leb] that

$$\exists C > 0 \text{ such that } \forall \tau \in \operatorname{Spec} \mathcal{B}\,, \quad \operatorname{Im} \tau \leq -\frac{1}{C} \operatorname{e}^{-C|\operatorname{Re} \tau|}\,.$$

Let us mention a result comparable to Theorem 1 in the framework of chaotic scattering obtained recently by Nonnenmacher and Zworski [NoZw], in the semiclassical setting. For a large class of Hamiltonians, including $P(\hbar) = -\hbar\Delta + V$ on $\mathbb{R}^d$ with V compactly supported, they were able to show a resonance-free region near the energy E:

$$\exists \delta, \gamma > 0 \text{ such that } \operatorname{Res}(P(\hbar)) \cap ([E - \delta, E + \delta] - \mathrm{i}[0, \gamma\hbar]) = \emptyset \text{ for } 0 < \hbar \leq \hbar_{\delta,\gamma}.$$

This holds provided that the hamiltonian flow Φ^t on the trapped set K_E at energy E is hyperbolic, and that the pressure of the unstable Jacobian with respect to the geodesic flow on K_E is strictly negative. We will adapt several techniques of [NoZw] to prove Theorem 1, some of them coming back to [Ana1, AnNo].

In a recent paper, Anantharaman [Ana2] studied the spectral deviations of Spec $\mathcal{B}$ with respect to the line of accumulation $\operatorname{Im} z = -\bar{a}$ appearing in (1.6). In the case of constant negative curvature, she obtained an upper bound for the number of modes with imaginary parts above $-\bar{a}$, and showed that for $\alpha \in [-\bar{a}, 0[$, there exists a function $H(\alpha)$ such that

(1.10)
$$\forall c > 0, \forall \varepsilon > 0, \quad \limsup_{\lambda \to \infty} \frac{\log \operatorname{Card}\{\tau_n : \operatorname{Re} \tau_n \in [\lambda - c, \lambda + c],\ \operatorname{Im} \tau_n \geq \alpha + \varepsilon\}}{\log \lambda} \leq H(\alpha).$$

$H(\alpha)$ is a dynamical quantity defined by

$$H(\alpha) = \sup\{h_{KS}(\mu),\ \mu \in \mathcal{M}_{\frac{1}{2}},\ \int a d\mu = -\alpha\}$$

where $\mathcal{M}_{\frac{1}{2}}$ denotes the set of Φ^t−invariant measures on S^*M, and h_{KS} stands for the Kolmogorov–Sinai entropy of μ. As a consequence of Theorem 1, the result of Anantharaman is not always optimal : $H(\alpha) \neq 0$ for $\alpha \in [-\bar{a}, 0[$, but if $\Pr(a^u) < 0$, there is no spectrum in a strip of finite width below the real axis, i.e. the lim sup in (1.10) vanishes for some $\alpha = \alpha(a) \neq 0$.

The operator $\mathcal{B}$ being non-selfadjoint, its eigenfunctions may fail to form a Riesz basis of $\mathcal{H}$. However, if a solution $\boldsymbol{u}$ of (1.3) has initial data sufficiently regular, it is still possible to expand it on eigenfunctions which eigenmodes are close to the real axis, up to an exponentially small error in time :

Theorem 2. (Eigenvalues expansion) *Let $\varepsilon > 0$ such that $\Pr(a^u) + \varepsilon < 0$, and $\kappa \geq \frac{d}{2}$. There exists $e_0(\varepsilon) > 0$, $n = n(\varepsilon) \in \mathbb{N}$ and a (finite) sequence $\tau_0, \dots, \tau_{n-1}$ of eigenvalues of $\mathcal{B}$ with*

$$\tau_j \in [-e_0(\varepsilon), e_0(\varepsilon)] + \mathrm{i}[\Pr(a^u) + \varepsilon, 0], \quad j \in [\![0, n-1]\!]\,,$$

such that for any solution $\boldsymbol{u}(t,x)$ of (1.3) with initial data $\boldsymbol{\omega} \in \mathcal{H}^\kappa$, we have

$$\boldsymbol{u}(t,x) = \sum_{j=0}^{n-1} \mathrm{e}^{-\mathrm{i}\, t\tau_j}\, \boldsymbol{u}_j(t,x) + \mathbf{r}_n(t,x)\,, \quad t > 0\,.$$

The functions $\boldsymbol{u}_j, r_n$ satisfy

$$\|\boldsymbol{u}_j(t,\cdot)\|_{\mathcal{H}} \leq C t^{m_j} \|\boldsymbol{\omega}\|_{\mathcal{H}} \quad \text{and} \quad \|\mathbf{r}_n(t,\cdot)\|_{\mathcal{H}} \leq C_\varepsilon\, \mathrm{e}^{t(\Pr(a^u)+\varepsilon)}\, \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa},$$

where m_j denotes the multiplicity of τ_j, the constants $C > 0$ depends only on M and a, while $C_\varepsilon > 0$ depending on M, a and ε.

A similar eigenvalues expansion can be found in [Hit], where no particular assumption on the curvature of M is made, however the geometric control must hold. Our last result deals with an exponential decay of the energy, which will be derived a consequence of the preceding theorem :

Theorem 3. (Exponential energy decay) *Let $\varepsilon > 0$, $(\tau_j)_{0 \leq j \leq n(\varepsilon)-1}$, κ and $\boldsymbol{u}$ as in Theorem 2. Set by convention $\tau_0 = 0$. The energy $E(u,t)$ satisfies*

$$E(u,t) \leq \left(\sum_{j=1}^{n-1} \mathrm{e}^{t\,\mathrm{Im}\,\tau_j}\, t^{m_j} C \|\boldsymbol{\omega}\|_{\mathcal{H}} + C_\varepsilon\, \mathrm{e}^{t(\Pr(a^u)+\varepsilon)}\, \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa} \right)^2$$

where m_j denotes the multiplicity of τ_j. The constants $C > 0$ depends only on M and a, while $C_\varepsilon > 0$ depending on M, a and ε. In particular, $\rho(\kappa) = 2\min(G, |\Pr(a^u) + \varepsilon|) > 0$.

Remark. In our setting, it may happen that geometric control does not hold, while $\Pr(a^u) < 0$. In this particular situation, it follows from [BLR] that we can not have an exponential energy decay uniformly for all Cauchy data in $\mathcal{H}$, where by uniform we mean that the constant C appearing in (1.7) does not depend on $\boldsymbol{u}$. However, if for $\kappa \geq \frac{d}{2}$ we look at $\rho(\kappa)$ instead of $\rho(0)$, our results show that we still have uniform exponential decay, namely $\rho(\kappa) > 0$ while $\rho(0) = 0$.

1.1. **Semiclassical reduction.** The main step yielding to Theorem (1) is more easily achieved when working in a semiclassical setting. From the eigenvalue equation (1.5), we are lead to study the equation

$$P(\tau)u = 0$$

where $\mathrm{Im}\,\tau = \mathcal{O}(1)$. To obtain a spectral gap below the real axis, we are lead to study eigenvalues with arbitrary large real parts since $\mathrm{Spec}\,\mathcal{B}$ is discrete. For this purpose, we introduce a semiclassical parameter $\hbar \in]0,1]$, and write the eigenvalues as

$$\tau = \frac{1}{\hbar} + \mathcal{O}(1).$$

If we let $\hbar$ go to 0, the eigenvalues τ we are interested in then satisfy $\tau\hbar \xrightarrow{\hbar \to 0} 1$. Putting $\tau = \frac{\lambda}{\hbar}$ and $z = \lambda^2/2$, we rewrite the stationary equation

$$\left(-\frac{\hbar^2 \Delta}{2} - z - \mathrm{i}\,\hbar q_z \right) u = 0\,, \quad q_z(x) = \sqrt{2z}\,a(x).$$

Equivalently, we write

$$(\mathcal{P}(z,\hbar)-z)u=0 \tag{1.11}$$

where $\mathcal{P}(z,\hbar)=-\frac{\hbar^2\Delta}{2}-\mathrm{i}\,\hbar q_z$. The parameter z plays the role of a complex eigenvalue of the non-selfadjoint quantum Hamiltonian $\mathcal{P}$. It is close to the "energy" $E=1/2$, while $\operatorname{Im} z$ is of order $\hbar$ and represents the "decay rate" of the mode. In order to recall these properties, we will often write

$$z=\frac{1}{2}+\hbar\zeta\,,\quad \zeta\in\mathbb{C}\ \text{ and }\ |\zeta|=\mathcal{O}(1). \tag{1.12}$$

In most of the following, we will deal with the semiclassical analysis of the non-selfadjoint Schrödinger operator $\mathcal{P}(z,\hbar)$ and the associated Schrödinger equation

$$\mathrm{i}\,\hbar\partial_t\Psi=\mathcal{P}(z,\hbar)\Psi \quad \text{ with }\ \|\Psi\|_{L^2}=1. \tag{1.13}$$

The basic facts and notations we will use from semiclassical analysis are recalled in Appendix A. The operator $\mathcal{P}$ has a principal symbol equal to $p(x,\xi)=\frac{1}{2}g_x(\xi,\xi)$, and a subprincipal symbol given by $-\mathrm{i}\,q_z$. Note that the classical Hamiltonian $p(x,\xi)$ generates the geodesic flow on the energy surface $p^{-1}(\frac{1}{2})=S^*M$. The properties of the geodesic flow on S^*M which will be useful to us are summarized in the next section, where is also given an alternative definition of the topological pressure more adapted to our purposes. We will denote the quantum propagator by

$$\mathcal{U}^t\equiv \mathrm{e}^{-\frac{\mathrm{i}\,t}{\hbar}\mathcal{P}},$$

so that if $\Psi\in L^2(M)$ satisfies (1.13), we have $\Psi(t)=\mathcal{U}^t\Psi(0)$. Using standard methods of semiclassical analysis, one can show that $\mathcal{U}^t$ is a Fourier integral operator (see [EvZw], chapter 10) associated with the symplectic diffeomorphism given by the geodesic flow Φ^t. Since we assumed that $a\geq 0$, it is true that $\|\mathcal{U}^t\|_{L^2\to L^2}\leq 1,\ \forall t\geq 0$.

Denote

$$\Sigma_{\frac{1}{2}}=\{z=\frac{1}{2}+\mathcal{O}(\hbar)\in\mathbb{C},\ \exists\Psi\in L^2(M),\ (\mathcal{P}(z,\hbar)-z)\Psi=0\}.$$

If $z\in\Sigma_{\frac{1}{2}}$ and Ψ is such that (1.11) holds, the semiclassical wave front set of Ψ satisfies

$$\mathrm{WF}_\hbar(\Psi)\subset S^*M.$$

This comes from the fact that Ψ is an eigenfunction associated with the eigenvalue $\frac{1}{2}$ of a pseudodifferential operator with principal symbol $p(x,\xi)=\frac{1}{2}g_x(\xi,\xi)$. Using these semiclassical settings, we will show the following key result :

Theorem 4. *Let $z\in\Sigma_{\frac{1}{2}}$, and $\varepsilon>0$ be such that $\Pr(a^u)+\varepsilon<0$. There exists $\hbar_0=\hbar_0(\varepsilon)$ such that*

$$\hbar\leq\hbar_0\ \Rightarrow\ \frac{\operatorname{Im} z}{\hbar}\leq \Pr(a^u)+\varepsilon.$$

From (1.12), we also notice that the above equation implies $\operatorname{Im}\tau\leq\Pr(a^u)+\varepsilon+\mathcal{O}(\hbar)$ since $\tau=\hbar^{-1}\sqrt{2z}$, and then $\operatorname{Im}\tau=\operatorname{Im}\zeta+\mathcal{O}(\hbar)$. It follows by rescaling that Theorem 4 is equivalent to Theorem 1.

2. Quantum dynamics and spectral gap

2.1. **Hyperbolic flow and topological pressure.** We call

$$\Phi^t=\mathrm{e}^{tH_p}:T^*M\to T^*M$$

the geodesic flow, where H_p is the Hamilton vector field of p. In local coordinates,

$$H_p \stackrel{\text{def}}{=} \sum_{i=1}^{d} \frac{\partial p}{\partial \xi_i}\partial_{x_i} - \frac{\partial p}{\partial x_i}\partial_{\xi_i} = \{p, \cdot\}$$

where the last equality refers to the Poisson bracket with respect to the canonical symplectic form $\omega = \sum_{i=1}^{d} d\xi_i \wedge dx_i$. Since M has strictly negative curvature, the flow generated by H_p on constant energy layers $\mathcal{E} = p^{-1}(E) \subset T^*M,\ E > 0$ has the Anosov property: for any $\rho \in \mathcal{E}$, the tangent space $T_\rho\mathcal{E}$ splits into flow, stable and unstable subspaces

$$T_\rho\mathcal{E} = \mathbb{R}H_p \oplus E^s(\rho) \oplus E^u(\rho)\,.$$

The spaces $E^s(\rho)$ and $E^u(\rho)$ are $d-1$ dimensional, and are stable under the flow map:

$$\forall t \in \mathbb{R},\quad d\Phi^t_\rho(E^s(\rho)) = E^s(\Phi^t(\rho)),\quad d\Phi^t_\rho(E^u(\rho)) = E^u(\Phi^t(\rho)).$$

Moreover, there exist $C, \lambda > 0$ such that

$$\begin{aligned} &i) && \|d\Phi^t_\rho(v)\| \le C\,\mathrm{e}^{-\lambda t}\,\|v\|, \quad \text{for all } v \in E^s(\rho),\ t \ge 0 \\ &ii) && \|d\Phi^{-t}_\rho(v)\| \le C\,\mathrm{e}^{-\lambda t}\,\|v\|, \quad \text{for all } v \in E^u(\rho),\ t \ge 0. \end{aligned} \tag{2.1}$$

One can show that there exist a metric on T^*M call the adapted metric, for which one can takes $C = 1$ in the preceding equations. At each point ρ, the spaces $E^u(\rho)$ are tangent to the unstable manifold $W^u(\rho)$, the set of points $\rho^u \in \mathcal{E}$ such that $d(\Phi^t(\rho^u), \Phi^t(\rho)) \xrightarrow{t\to-\infty} 0$ where d is the distance induced from the adapted metric. Similarly, $E^s(\rho)$ is tangent to the stable manifold $W^s(\rho)$, the set of points ρ^s such that $d(\Phi^t(\rho^s), \Phi^t(\rho)) \xrightarrow{t\to+\infty} 0$.

The adapted metric induces a the volum form Ω_ρ on any d dimensional subspace of $T(T^*_\rho M)$. Using Ω_ρ, we now define the unstable Jacobian at ρ for time t. Let us define the weak-stable and weak-unstable subspaces at ρ by

$$E^{s,0}(\rho) = E^s(\rho) \oplus \mathbb{R}H_p\,,\quad E^{u,0}(\rho) = E^u(\rho) \oplus \mathbb{R}H_p.$$

We set

$$J^u_t(\rho) = \det d\Phi^{-t}|_{E^{u,0}(\Phi^t(\rho))} = \frac{\Omega_\rho(d\Phi^{-T}v_1 \wedge \cdots \wedge d\Phi^{-t}v_d)}{\Omega_{\Phi^t(\rho)}(v_1 \wedge \cdots \wedge v_d)}\,,\qquad J^u(\rho) \stackrel{\text{def}}{=} J^u_1(\rho),$$

where $(v_1, \ldots, v_d)$ can be any basis of $E^{u,0}(\rho)$. While we do not necessarily have $J^u(\rho) < 1$, it is true that $J^u_t(\rho)$ decays exponentially as $t \to +\infty$.

The definition of the topological pressure of the geodesic flow given in the introduction, although quite straighforward to state, is not really suitable for our purposes. The alternative definition of the pressure we will work with is based on refined covers of S^*M, and can be stated as follows. For $\delta > 0$, let $\mathcal{E}^\delta = p^{-1}[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ be a thin neighbourhood of the constant energy surface $p^{-1}(\frac{1}{2})$, and $V = \{V_\alpha\}_{\alpha\in I}$ an open cover of $\mathcal{E}^\delta$. In what follows, we shall always choose $\delta < 1/2$. For $T \in \mathbb{N}^*$, we define the refined cover $V^{(T)}$, made of the sets

$$V_\beta = \bigcap_{k=0}^{T-1} \Phi^{-k}(V_{b_k})\,,\quad \beta = b_0 b_1 \ldots b_{T-1} \in I^T.$$

It will be useful to coarse-grain any continuous function f on $\mathcal{E}^\delta$ with respect to $V^{(T)}$ by setting

$$\langle f\rangle_{T,\beta} = \sup_{\rho\in V_\beta} \sum_{i=0}^{T-1} f \circ \Phi^i(\rho).$$

One then define

$$Z_T(V,f) = \inf_{B_T} \left\{ \sum_{\beta\in B_T} \exp(\langle f\rangle_{T,\beta}) \;:\; B_T \subset I^T\,,\ \mathcal{E}^\delta \subset \bigcup_{\beta\in B_T} V_\beta \right\}.$$

The topological pressure of f with respect to the geodesic flow on $\mathcal{E}^\delta$ is defined by :

$$\mathrm{Pr}^\delta(f) = \lim_{\operatorname{diam} V\to 0} \lim_{T\to\infty} \frac{1}{T}\log Z_T(V,f)\,.$$

The pressure on the unit tangent bundle S^*M is simply obtained by continuity, taking the limit $\mathrm{Pr}(f) = \lim_{\delta\to 0}\mathrm{Pr}^\delta(f)$. To make the above limits easier to work with, we now take $f = a^u$ and fix $\varepsilon > 0$ such that $\mathrm{Pr}(a^u)+\varepsilon < 0$. Then, we choose the width of the energy layer $\delta \in]0,1[$ sufficiently small such that $|\,\mathrm{Pr}(a^u) - \mathrm{Pr}^\delta(a^u)| \leq \varepsilon/2$. Given a cover $\mathcal{V} = \{\mathcal{V}_\alpha\}_{\alpha\in\mathcal{A}}$ (of arbitrary small diameter), there exist a time t_0 depending on the cover $\mathcal{V}$ such that

$$\left|\frac{1}{t_0}\log Z_{t_0}(\mathcal{V},a^u) - \mathrm{Pr}^\delta(a^u)\right| \leq \frac{\varepsilon}{2}\,.$$

Hence there is a subset of t_0-strings $\mathcal{B}_{t_0} \subset \mathcal{A}^{t_0}$ such that $\{\mathcal{V}_\alpha\}_{\alpha\in\mathcal{B}_{t_0}}$ is an open cover of $\mathcal{E}^\delta$ and satisfies

$$\sum_{\beta\in\mathcal{B}_{t_0}} \exp(\langle a^u\rangle_{t_0,\beta}) \leq \exp\left(t_0(\mathrm{Pr}^\delta(a^u)+\frac{\varepsilon}{2})\right) \leq \exp\left(t_0(\mathrm{Pr}(a^u)+\varepsilon)\right)\,. \tag{2.2}$$

For convenience, we denote by $\{\mathcal{W}_\beta\}_{\beta\in\mathcal{B}_{t_0}} \equiv \{\mathcal{V}_\beta\}_{\beta\in\mathcal{B}_{t_0}}$ the sub-cover of $\mathcal{V}^{(t_0)}$ such that (2.2) holds. Note that in this case, the diameter of $\mathcal{V}$, t_0 and then $\mathcal{W}$ depends on ε.

2.2. **Discrete time evolution.** Let $\{\varphi_\beta\}_{\beta\in\mathcal{B}_{t_0}}$ be a partition of unity adapted to $\mathcal{W}$, so that its Weyl quantization $\varphi_\beta^w \stackrel{\mathrm{def}}{=} \Pi_\beta$ (see Appendix A) satisfy

$$\mathrm{WF}_\hbar(\Pi_\beta) \subset \mathcal{E}^\delta, \quad \Pi_\beta^* = \Pi_\beta, \quad \sum_\beta \Pi_\beta = 1\!\!1 \quad \text{microlocally near } \mathcal{E}^{\delta/2}\,.$$

We will also consider a partition of unity $\{\tilde\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ adapted to the cover $\mathcal{V}$, and its Weyl quantization $\tilde\Pi \stackrel{\mathrm{def}}{=} \tilde\varphi^w$. In what follows, we will be interested in the propagator $\mathcal{U}^{Nt_0+1}$, and

$$N = T\log\hbar^{-1},\ T>0\,.$$

It is important to note that T can be arbitrary large, but is fixed with respect to $\hbar$. The propagator $\mathcal{U}^{Nt_0}$ is decomposed by inserting $\sum_{\beta\in\mathcal{B}_{t_0}}\Pi_\beta$ at each time step of length t_0. Setting first $\mathcal{U}_\beta = \mathcal{U}^{t_0}\Pi_\beta$, we have (microlocally near $\mathcal{E}^{\delta/2}$) the equality $\mathcal{U}^{t_0} = \sum_\beta \mathcal{U}_\beta$, and then

$$\mathcal{U}^{Nt_0} = \sum_{\beta_1,\beta_2,\ldots,\beta_N\in\mathcal{B}_{t_0}} \mathcal{U}_{\beta_N}\ldots\mathcal{U}_{\beta_1}\,, \quad \text{near } \mathcal{E}^{\delta/2}. \tag{2.3}$$

2.3. **Proof of Theorem 4.** We begin by choosing $\chi \in C_0^\infty(T^*M)$ such that $\operatorname{supp}\chi \Subset \mathcal{E}^\delta$ and $\chi \equiv 1$ on $\mathcal{E}^{\delta/4}$, and considering $\mathrm{Op}_\hbar(\chi)$. Applying the Cauchy-Schwartz inequality, we get immediately

$$\|\mathcal{U}^{Nt_0+1}\,\mathrm{Op}_\hbar(\chi)\| \leq \sum_{\beta_1,\beta_2,\ldots,\beta_N\in\mathcal{B}_{t_0}^N} \|\mathcal{U}_{\beta_N}\ldots\mathcal{U}_{\beta_1}\mathcal{U}^1\,\mathrm{Op}_\hbar(\chi)\| + \mathcal{O}_{L^2\to L^2}(\hbar^\infty). \tag{2.4}$$

Unless otherwise stated, the norms $\|\cdot\|$ always refer to $\|\cdot\|_{L^2\to L^2}$ or $\|\cdot\|_{L^2}$, according to the context. The proof of Theorem (4) relies on the following intermediate result, proven much later in Section 4.

Proposition 5. (Hyperbolic dispersion estimate) *Let $\varepsilon > 0$, and $\delta, \mathcal{V}, t_0$ be as in Section 2.1. For $N = Tt_0 \log \hbar^{-1}$, $T > 0$, take a sequence $\beta_1, \dots \beta_N$ and $\mathcal{W}_1, \dots, \mathcal{W}_{\beta_N}$ the associated open sets of the refined cover $\mathcal{W}$. Finally , let $\mathrm{Op}_\hbar(\chi)$ be as above. There exists a constant $C > 0$ and $\hbar_0(\varepsilon) \in]0,1[$ such that*

$$\hbar \leq \hbar_0 \;\Rightarrow\; \|\mathcal{U}^{t_0}\Pi_{\beta_N} \dots \mathcal{U}^{t_0}\Pi_{\beta_1}\mathcal{U}^1 \mathrm{Op}_\hbar(\chi)\| \leq C\hbar^{-d/2} \prod_{j=1}^{N} \mathrm{e}^{\langle a^u \rangle_{t_0,\beta_j}}$$

where $\langle a^u \rangle_{t_0,\beta} = \sup_{\rho\in\mathcal{W}_\beta} \sum_{j=0}^{t_0-1} a^u \circ \Phi^j(\rho)$. The constant C only depends on the manifold M.

We also state the following crucial consequence :

Corollary 6. *Take $\varepsilon > 0$ such that $\mathrm{Pr}(a^u) + \varepsilon < 0$. There exists $C > 0$ and $\hbar_0(\varepsilon) \in]0,1[$ such that*

$$\hbar \leq \hbar_0 \Rightarrow \|\mathcal{U}^{Nt_0+1} \mathrm{Op}_\hbar(\chi)\| \leq C\hbar^{-\frac{d}{2}} \mathrm{e}^{Nt_0(\mathrm{Pr}(a^u)+\varepsilon)}$$

The constant C only depends on M.

Proof. Given $\varepsilon > 0$, we choose $\delta, \mathcal{V}, t_0, \mathcal{W}$ as in the preceding proposition. Using (2.4), we then have

$$\begin{aligned} \|\mathcal{U}^{Nt_0+1} \mathrm{Op}_\hbar(\chi)\| &\leq C\hbar^{-\frac{d}{2}} \sum_{\beta_1\dots\beta_N \in \mathcal{B}_{t_0}^N} \left(\prod_{j=1}^{N} \mathrm{e}^{\langle a^u \rangle_{t_0,\beta_j}} + \mathcal{O}(\hbar^\infty) \right) \\ &\leq C\hbar^{-\frac{d}{2}} \left(\sum_{\beta\in\mathcal{B}_{t_0}} \mathrm{e}^{\langle a^u \rangle_{t_0,\beta}} \right)^N + \mathcal{O}(\hbar^\infty). \end{aligned}$$

To get the second line, notice that the number of terms in the sum is of order $(\mathrm{Card}\,\mathcal{B}_{t_0})^N = \hbar^{-T\log \mathrm{Card}\,\mathcal{B}_{t_0}}$. From our choice of ε and δ, we can use (2.2), and for $\hbar$ small enough, rewrite this equation as

$$\begin{aligned} \|\mathcal{U}^{Nt_0+1} \mathrm{Op}_\hbar(\chi)\| &\leq C\hbar^{-\frac{d}{2}} \mathrm{e}^{Nt_0(\mathrm{Pr}^\delta(a^u)+\varepsilon/2)} \\ &\leq C\hbar^{-\frac{d}{2}} \mathrm{e}^{Nt_0(\mathrm{Pr}(a^u)+\varepsilon)} \end{aligned}$$

where $C > 0$ only depends on the manifold M. □

Let us show how this result implies Theorem 4. We assume that Ψ satisfies (1.11), and therefore $\|\mathcal{U}^{Nt_0+1}\Psi\| = \mathrm{e}^{\frac{(Nt_0+1)\,\mathrm{Im}\, z}{\hbar}}$. Notice also that we have

$$\mathrm{Op}_\hbar(\chi)\Psi = \Psi + \mathcal{O}(\hbar^\infty)$$

since $\mathrm{WF}_\hbar(\Psi) \subset S^*M$, and then

$$\|\mathcal{U}^{Nt_0+1} \mathrm{Op}_\hbar(\chi)\Psi\| = \|\mathcal{U}^{Nt_0+1}\Psi\| + \mathcal{O}(\hbar^\infty) = \mathrm{e}^{(Nt_0+1)\,\mathrm{Im}\, z} + \mathcal{O}(\hbar^\infty)\,.$$

It follows from the corollary that

$$\mathrm{e}^{\frac{Nt_0+1}{\hbar}\,\mathrm{Im}\, z} \leq C\hbar^{-\frac{d}{2}} \mathrm{e}^{Nt_0(\mathrm{Pr}(q^u)+\varepsilon)} + \mathcal{O}(\hbar^m)\,,$$

where m can be arbitrary large. Taking the logarithm, this yields to

$$\frac{\operatorname{Im} z}{\hbar} \leq \frac{\log C}{Nt_0} - \frac{d}{2Nt_0}\log\hbar + \Pr(q^u) + \varepsilon + \mathcal{O}(\frac{1}{Nt_0})\,.$$

But given $\varepsilon > 0$, we can take $N = T\log\hbar^{-1}$ with T arbitrary. Hence there is $\hbar_0(\varepsilon) \in]0,1[$ and T sufficiently large, such that

$$\hbar \leq \hbar_0(\varepsilon) \Rightarrow \frac{\operatorname{Im} z}{\hbar} \leq \Pr(q^u) + 2\varepsilon.$$

Since the parameter ε can be chosen as small as wished, this proves Theorem 4.

3. Eigenvalues expansion and energy decay

3.1. **Resolvent estimates.** To show the exponential decay of the energy, we follow a standard route from resolvent estimates in a strip around the real axis. Let us denote

$$Q(z,\hbar) = -\frac{\hbar^2}{2}\Delta - z - \mathrm{i}\,\hbar\sqrt{2z}a(x) = \mathcal{P}(z,\hbar) - z.$$

The following proposition establish a resolvent estimate in a strip of width $|\Pr(a^u) + \varepsilon|$ below the real axis in the semiclassical limit. This is the main step toward Theorems 2 and 3, see also [NoZw2] for comparable resolvent estimates in the chaotic scattering situation :

Proposition 7. *Let $\varepsilon > 0$. Choose $\gamma < 0$ such that*

$$\Pr(a^u) + \varepsilon < \gamma < 0\,,$$

and $z = \frac{1}{2} + \hbar\zeta$, with $|\zeta| = \mathcal{O}(1)$ satisfying

$$\gamma \leq \operatorname{Im}\zeta \leq 0\,.$$

There exists $\hbar_0(\varepsilon) > 0$, $C_\varepsilon > 0$ depending on M, a and ε such that

$$\hbar \leq \hbar_0(\varepsilon) \;\;\Rightarrow\;\; \|Q(z,\hbar)^{-1}\|_{L^2\to L^2} \leq C_\varepsilon \hbar^{-1+c_0\operatorname{Im}\zeta}\log\hbar^{-1}$$

where $c_0 = \frac{d}{2|\Pr(q^u)+\varepsilon|}$.

Proof. Given $\varepsilon > 0$, we fix $\hbar_0(\varepsilon)$ so that Corollary 6 holds. Finally, we define

$$\Pr(a^u)^+ = \Pr(a^u) + \varepsilon.$$

In order to bound $Q(z,\hbar)^{-1}$, we proceed in two steps, by finding two operators which approximate Q^{-1}: one on the energy surface $\mathcal{E}^\delta$, the other outside $\mathcal{E}^\delta$. Let χ be as in Section 2, and choose also Let $\tilde\chi \in C_0^\infty(T^*M)$ with $\operatorname{supp}\tilde\chi \Subset \operatorname{supp}\chi$, such that we also have $\tilde\chi = 1$ near S^*M. We first look for an operator $A_0 = A_0(z,\hbar)$ such that

$$QA_0 = (1 - \mathrm{Op}_\hbar(\chi)) + \mathcal{O}_{L^2\to L^2}(\hbar^\infty).$$

For this, consider $Q(z,\hbar) + \mathrm{i}\,\mathrm{Op}_\hbar(\tilde\chi) \stackrel{\text{def}}{=} Q_0(z,\hbar)$. Because of the property of $\tilde\chi$, the operator Q_0 is elliptic. Hence, there is an operator $\tilde A_0$, uniformly bounded in $L^2(M)$, such that

$$Q_0\tilde A_0 = \mathrm{Id} + \mathcal{O}_{L^2\to L^2}(\hbar^\infty).$$

The operator A_0 we are looking for is obtained by taking $A_0 = \tilde A_0(1 - \mathrm{Op}_\hbar(\chi))$. Indeed,

$$\begin{aligned} Q(z,\hbar)A_0(z,\hbar) &= 1 - \mathrm{Op}_\hbar(\chi) - \mathrm{i}\,\mathrm{Op}_\hbar(\tilde\chi)\tilde A_0(1 - \mathrm{Op}_\hbar(\chi)) + \mathcal{O}_{L^2\to L^2}(\hbar^\infty) \\ &= 1 - \mathrm{Op}_\hbar(\chi) + \mathcal{O}_{L^2\to L^2}(\hbar^\infty) \end{aligned}$$

since $\tilde\chi$ and $1-\chi$ have disjoints supports by construction.

We now look for the solution on $\mathcal{E}^\delta$. From Corollary 6, we have an exponential decay of the propagator $\mathcal{U}^{Nt_0}$ if N becomes large. To use this information, we set

$$A_1(z,\hbar,T_\hbar)=\frac{\mathrm{i}}{\hbar}\int_0^{T_\hbar}\mathcal{U}^t\,\mathrm{e}^{\frac{\mathrm{i}\,t}{\hbar}z}\,\mathrm{Op}_\hbar(\chi)\,dt$$

where $T_\hbar$ has to be adjusted. Hence,

$$Q(z,\hbar)A_1(z,\hbar,T_\hbar)=\mathrm{Op}_\hbar(\chi)-\mathcal{U}^t\,\mathrm{e}^{\frac{\mathrm{i}\,t}{\hbar}z}\,|_{t=T_\hbar}\,\mathrm{Op}_\hbar(\chi)=\mathrm{Op}_\hbar(\chi)+R_1\,.$$

Since $\operatorname{Im} z/\hbar=\operatorname{Im}\zeta=\mathcal{O}(1)$, we have

$$\|R_1\|=\mathrm{e}^{-T_\hbar\operatorname{Im}\zeta}\,\|\mathcal{U}^{T_\hbar}\,\mathrm{Op}_\hbar(\chi)\|. \tag{3.1}$$

From Corollary 6, we know that for $N=Tt_0\log\hbar^{-1}$ with $\hbar\leq\hbar_0$,

$$\|\mathcal{U}^{Nt_0+1}\,\mathrm{Op}_\hbar(\chi)\|\leq C\hbar^{-\frac{d}{2}}\,\mathrm{e}^{Nt_0\Pr(a^u)^+}\,.$$

We observe that this bound is useful only if does not diverge as $\hbar\to 0$, which is the case if

$$T\geq\frac{d}{2t_0|\Pr(a^u)^+|}\stackrel{\text{def}}{=}T_0.$$

Let us define

$$T_\hbar^0=T_0t_0\log\hbar^{-1}+1=\frac{d}{2|\Pr(a^u)^+|}\log\hbar^{-1}+1\,. \tag{3.2}$$

If $T_\hbar=T\log\hbar^{-1}t_0+1$, with $T>T_0$ chosen large enough, we find

$$\|R_1\|\quad\leq\quad C\hbar^{Tt_0\gamma}\hbar^{-\frac{d}{2}-Tt_0\Pr(a^u)^+}=\mathcal{O}(\hbar^m)$$

with $m=m(T_\hbar)\geq 0$, since $\gamma-\Pr(a^u)^+>0$. Consequently, there is $T_1=T_1(\varepsilon)>0$ such that $T_\hbar^1=T_1t_0\log\hbar^{-1}+1$ satisfies $m(T_\hbar^1)=0$. This means that for $T_\hbar\geq T_\hbar^1$, we have

$$Q(z,\hbar)(A_0(z,\hbar)+A_1(z,\hbar,T_h))=1+\mathcal{O}_{L^2\to L^2}(1),$$

in other words, A_0+A_1 is "close" to the resolvent Q^{-1} . Hence, we impose now $T_\hbar\geq T_\hbar^1$, and evaluate the norms of A_0 and A_1. By construction, $\|A_0\|=\mathcal{O}(1)$. For A_1, we have to estimate an integral of the form

$$I_{T_\hbar}=\int_0^{T_\hbar}\mathrm{e}^{-t\operatorname{Im}\zeta}\,\|\mathcal{U}^t\,\mathrm{Op}_\hbar(\chi)\|dt.$$

Let us split the integral according to $T_\hbar^0$, and use the decay of $\mathcal{U}^t\,\mathrm{Op}_\hbar(\chi)$ for $t\geq T_\hbar^0$:

$$\begin{aligned}|I_{T_\hbar}|\quad&\leq\quad T_\hbar^0\,\mathrm{e}^{-T_\hbar^0\operatorname{Im}\zeta}+C\hbar^{-\frac{d}{2}}\int_{T_\hbar^0}^\infty\mathrm{e}^{-t\operatorname{Im}\zeta}\,\mathrm{e}^{(t-1)\Pr(a^u)^+}\,dt\\&\leq\quad T_\hbar^0\,\mathrm{e}^{-T_\hbar^0\operatorname{Im}\zeta}(1+C_\varepsilon\hbar^{-\frac{d}{2}}\,\mathrm{e}^{(T_\hbar^0-1)\Pr(a^u)^+})\\&\leq\quad C_\varepsilon T_\hbar^0\,\mathrm{e}^{-T_\hbar^0\operatorname{Im}\zeta}\ .\end{aligned}$$

Using (3.2), this gives

$$\|A_1(z,\hbar,T_\hbar)\|\leq C_\varepsilon\hbar^{-1+c_0\operatorname{Im}\zeta}\log\hbar^{-1}$$

where $C_\varepsilon>0$ depends now on M, a and ε while

$$c_0=\frac{d}{2|\Pr(a^u)^+|}. \tag{3.3}$$

□

We now translate these results obtained in the semiclassical settings in terms of τ. Recall

$$P(\tau) = -\Delta - \tau^2 - 2\,\mathrm{i}\, a\tau \equiv \frac{1}{\hbar^2} Q(z,\hbar).$$

and set $R(\tau) \overset{\text{def}}{=} P(\tau)^{-1}$. The operator $R(\tau)$ is directly related to the resolvent $(\tau - \mathcal{B})^{-1}$: a straightforward computation shows that

$$(\tau - \mathcal{B})^{-1} = \begin{pmatrix} R(\tau)(-2\,\mathrm{i}\, a - \tau) & -R(\tau) \\ R(\tau)(2\,\mathrm{i}\, a\tau - \tau^2) & -R(\tau)\tau \end{pmatrix}.$$

Proposition 8. *Let $\varepsilon > 0$ be such that $\Pr(a^u) + \varepsilon < 0$, and $\gamma < 0$ satisfying*

$$\Pr(a^u) + \varepsilon < \gamma < 0.$$

Let $\tau \in \mathbb{C} \setminus \operatorname{Spec} \mathcal{B}$ be such that $\gamma \leq \operatorname{Im} \tau < 0$. Set also $\langle \tau \rangle = (1 + |\tau|^2)^{\frac{1}{2}}$. There exists a constant $C > 0$ depending on M, a and ε such that for any $\kappa \geq d/2$, we have

$$\begin{aligned}
&(i) && \|R(\tau)\|_{L^2 \to L^2} \leq C_\varepsilon \langle \tau \rangle^{-1-c_0\gamma} \log\langle \tau \rangle \\
&(ii) && \|R(\tau)\|_{L^2 \to H^2} \leq C_\varepsilon \langle \tau \rangle^{1-c_0\gamma} \log\langle \tau \rangle \\
&(iii) && \|R(\tau)\|_{H^\kappa \to H^1} \leq C_\varepsilon \\
&(iv) && \|\tau R(\tau)\|_{H^\kappa \to H^0} \leq C_\varepsilon\,.
\end{aligned}$$

Proof. (i) follows directly from rescaling the statements of the preceding proposition. For (ii), observe that

$$\|R(\tau)u\|_{H^2} \leq C(\|R(\tau)u\|_{L^2} + \|\Delta R(\tau)u\|_{L^2})\,,\ C > 0.$$

But

$$\|\Delta R(\tau)u\|_{L^2} \leq \|u\|_{L^2} + |\tau^2 + 2\tau\,\mathrm{i}\, a|\|R(\tau)u\|_{L^2}\,,$$

so using (i), we get

$$\|R(\tau)u\|_{H^2} \leq C\left((1 + |\tau^2 + 2\,\mathrm{i}\, a\tau|)\|R(\tau)u\|_{L^2} + \|u\|_{L^2}\right) \leq C_\varepsilon \langle\tau\rangle^{1-c_0\gamma} \log\langle\tau\rangle \|u\|_{L^2}\,.$$

To arrive at (iii), we start from the following classical consequence of the Hölder inequality :

$$\|R(\tau)u\|^2_{H^{1-s}} \leq \|R(\tau)u\|^{1-s}_{H^2} \|R(\tau)u\|^{1+s}_{L^2}\,, \quad s > 0. \tag{3.4}$$

From (i) and (ii), we obtain

$$\|R(\tau)u\|_{H^{1-s}} \leq C_\varepsilon \langle\tau\rangle^{-(\gamma c_0 + s)} \log\langle\tau\rangle \|u\|_{L^2}\,.$$

If we choose $s > -\gamma c_0$, we get $\|R(\tau)\|_{H^0 \to H^{1-s}} \leq C_\varepsilon$. Hence, for any $s' \geq 0$ we have

$$\|R(\tau)\|_{H^{s'} \to H^{s'+1-s}} \leq C_\varepsilon\,.$$

Taking $s' = s$ shows (iii), where we must have $\kappa > -\gamma c_0$. In view of (3.3), and the fact that $\gamma \geq \Pr(a^u) + \tilde{\varepsilon}$, this condition is satisfied as soon as $\kappa \geq d/2$. The last equation (iv) is derived as (iii), by considering

$$\|\tau R(\tau)u\|^2_{H^{1-s}} \leq |\tau|^2 \|R(\tau)u\|^{1-s}_{H^2} \|R(\tau)u\|^{1+s}_{L^2}\,, \quad s > 0,$$

and choosing s so that $\|\tau R(\tau)\|^2_{H^0 \to H^{1-s}} \leq C_\varepsilon$. $\square$

3.2. **Eigenvalues expansion.** We now prove Theorem 2. Let us fix $\varepsilon > 0$ so that $\Pr(a^u) + \varepsilon < 0$. From Theorem 1 we know that

$$\mathrm{Card}\,(\mathrm{Spec}\,\mathcal{B} \cap (\mathbb{R} + \mathrm{i}[\Pr(a^u) + \varepsilon, 0])) \stackrel{\mathrm{def}}{=} n(\varepsilon) < \infty.$$

Hence there is $e_0(\varepsilon) > 0$ such that $\mathrm{Spec}\,\mathcal{B} \cap (\mathbb{R} + \mathrm{i}[\Pr(a^u) + \varepsilon]) \subset \Omega$, where

$$\Omega = \Omega(\varepsilon) = [-e_0, e_0] + \mathrm{i}[\Pr(a^u) + \varepsilon, 0].$$

We then call $\{\tau_0, \dots, \tau_{n(\varepsilon)-1}\} = \mathrm{Spec}\,\mathcal{B} \cap \Omega$, and set by convention $\tau_0 = 0$. We define as above $\Pr(a^u)^+ = \Pr(a^u) + \varepsilon$. Since we look at the eigenvalues $\tau \in \Omega$, let us introduce the spectral projectors on the generalized eigenspace E_j for $j \in [\![0, n-1]\!]$:

$$\Pi_j = \frac{1}{2\,\mathrm{i}\,\pi} \oint_{\gamma_j} (\tau - \mathcal{B})^{-1} d\tau\,, \quad \Pi_j \in \mathcal{L}(\mathcal{H}, D(\mathcal{B}^\infty)),$$

where γ_j are small circles centered in τ_j. We also denote by

$$\Pi = \sum_{j=0}^{n} \Pi_j$$

the spectral projection onto $\bigoplus_{j=0}^{n} E_j$. We call E_0 the eigenspace corresponding to the eigenvalue $\tau_0 = 0$. It can be shown [Leb] that E_0 is one dimensional over $\mathbb{C}$ and spanned by $(1,0)$, so

$$\Pi_0 \boldsymbol{\omega} = (c(\boldsymbol{\omega}), 0) \quad \text{with} \;\; c(\boldsymbol{\omega}) \in \mathbb{C}.$$

Let now $\boldsymbol{\omega} = (\omega_0, \omega_1)$ be in $\mathcal{H}^\kappa$. Near a pole τ_j of $(\tau - \mathcal{B})^{-1}$, we have

$$(\tau - \mathcal{B})^{-1} = \frac{\Pi_j}{\tau - \tau_j} + \sum_{k=2}^{m_j} \frac{(\mathcal{B} - \tau_j)^{k-1} \Pi_j}{(\tau - \tau_j)^k} + H_j(\tau)$$

where H_j is an operator depending holomorphically on τ in a neighbourhood of τ_j, and m_j is the multiplicity of τ_j. Since $\Pi \in \mathcal{L}(\mathcal{H}, D(\mathcal{B}^\infty))$, we have the following integral representation of $\mathrm{e}^{-\mathrm{i}\,t\mathcal{B}}\,\Pi\boldsymbol{\omega}$, with absolute convergence in $\mathcal{H}$:

$$\mathrm{e}^{-\mathrm{i}\,t\mathcal{B}}\,\Pi\boldsymbol{\omega} = \frac{1}{2\,\mathrm{i}\,\pi} \int_{-\infty+\mathrm{i}\,\alpha}^{+\infty+\mathrm{i}\,\alpha} \mathrm{e}^{-\mathrm{i}\,t\tau} (\tau - \mathcal{B})^{-1} \Pi\boldsymbol{\omega}\, d\tau\,, \quad t > 0,\; \alpha > 0. \tag{3.5}$$

The integrand in the right hand side has poles located at τ_j, $j \in [\![0, n-1]\!]$, so that

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\,t\mathcal{B}}\,\Pi\boldsymbol{\omega} &= \sum_j \frac{1}{2\,\mathrm{i}\,\pi} \oint_{\gamma_j} \mathrm{e}^{-\mathrm{i}\,t\tau} \frac{\Pi_j}{\tau - \tau_j} \boldsymbol{\omega}\, d\tau \\ &= \sum_j \mathrm{e}^{-\mathrm{i}\,t\tau_j}\, p_{\tau_j}(t)\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} \sum_j \mathrm{e}^{-\mathrm{i}\,t\tau_j}\, \boldsymbol{u}_j(t)\,. \end{aligned}$$

The operators $p_{\tau_j}(t)$ appearing in the residues are polynomials in t, with degree at most m_j, taking their values in $\mathcal{L}(\mathcal{H}, D(\mathcal{B}^\infty))$. It follows that for some $C > 0$ depending only on M and a,

$$\|\boldsymbol{u}_j(t)\|_{\mathcal{H}} \le C t^{m_j} \|\boldsymbol{\omega}\|_{\mathcal{H}}\,.$$

The remainder term appearing in Theorem 2 is now identified :

$$\mathbf{r}_n(t) = \mathrm{e}^{-\mathrm{i}\,t\mathcal{B}}(1 - \Pi)\boldsymbol{\omega}\,.$$

To conclude the proof, we have therefore to evaluate $\|\mathbf{r}_n\|_{\mathcal{H}}$. To do so, we will use in a crucial way the resolvent bounds below the real axis that we have obtained in the preceding section. We consider the solution $u(t,x)$ of (1.1) with initial data $\boldsymbol{u} = (u_0, u_1) = (1 - \Pi)\boldsymbol{\omega}$,

with $\boldsymbol{\omega} \in \mathcal{H}^\kappa$, $\kappa \geq d/2$. Let us define $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, such that $\chi = 0$ for $t \leq 0$ and $\chi = 1$ for $t \geq 1$. If we set $v = \chi u$, we have

$$(\partial_t^2 - \Delta + 2a\partial_t)v = g_1 \tag{3.6}$$

where

$$g_1 = \chi'' u + 2\chi' \partial_t u + 2a\chi' u\,. \tag{3.7}$$

Note also that $\operatorname{supp} g_1 \subset [0,1] \times M$, and $v(t) = 0$ for $t \leq 0$. Let us denote the inverse Fourier transform in time by

$$\mathcal{F}_{t\to -\tau} : u \mapsto \check{u}(\tau) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\, t\tau}\, u(t) dt.$$

Applying $\mathcal{F}_{t\to-\tau}$ (in the distributional sense) to both sides of (3.6) yields to

$$P(\tau)\check{v}(\tau, x) = \check{g}_1(\tau, x)\,.$$

We then remark that $R(\tau)\check{g}_1(\tau, x)$ is the first component of

$$\mathrm{i}(\tau - \mathcal{B})^{-1}\mathcal{F}_{t\to-\tau}\left(\chi'(t)(u, \mathrm{i}\,\partial_t u)\right).$$

From the properties of Π, it is clear that the operator $(\tau - \mathcal{B})^{-1}(1 - \Pi)$ depends holomorphically on τ in the half-plane $\operatorname{Im}\tau \geq \Pr(a^u)^+$. From $(u, \mathrm{i}\,\partial_t u) = \mathrm{e}^{-\mathrm{i}\, t\mathcal{B}}(1 - \Pi)\boldsymbol{\omega}$, we then conclude that $\mathrm{i}(\tau - \mathcal{B})^{-1}\mathcal{F}_{t\to-\tau}\left(\chi'(t)(u, \mathrm{i}\,\partial_t u)\right)$ depends also holomorphically on τ in the half plane $\operatorname{Im}\tau \geq \Pr(a^u)^+$. Hence $\check{v}(\tau, x) = R(\tau)\check{g}_1(\tau, x)$ and an application of the Parseval formula yields to

$$\begin{aligned}
\|\,\mathrm{e}^{-t\Pr(a^u)^+}\, v(t,x)\|_{L^2(\mathbb{R}_+,H^1)} &= \|\check{v}(\tau + \mathrm{i}\Pr(a^u)^+\|_{L^2(\mathbb{R},H^1)} \\
&= \|R(\tau + \mathrm{i}\Pr(a^u)^+)\check{g}_1(\tau + \mathrm{i}\Pr(a^u)^+, x)\|_{L^2(\mathbb{R},H^1)} \\
&\leq C_\varepsilon \|\check{g}_1(\tau + \mathrm{i}\Pr(a^u)^+, x)\|_{L^2(\mathbb{R},H^\kappa)} \\
&\leq C_\varepsilon \|g_1(t,x)\|_{L^2(\mathbb{R}_+,H^\kappa)}\,.
\end{aligned}$$

where we have used Proposition 8. The term appearing in the last line can in fact be controlled by the initial data. From (3.7), we have

$$\|g_1\|_{L^2(\mathbb{R}_+;H^\kappa)} \leq C\left(\|u\|_{L^2([0,1];H^\kappa)} + \|\partial_t u\|_{L^2([0,1];H^\kappa)}\right)\,. \tag{3.8}$$

A direct computation shows

$$\partial_t \|u\|^2_{\mathcal{H}^\kappa} \leq C(\|u\|^2_{\mathcal{H}^\kappa} + \|\partial_t u\|^2_{H^\kappa} + \|\nabla u\|^2_{H^\kappa})\,.$$

The Gronwall inequality for $t \in [0,1]$ gives

$$\begin{aligned}
\|u(t,\cdot)\|^2_{H^\kappa} &\leq C\left(\|u(0,\cdot)\|^2_{H^\kappa} + \int_0^t (\|\partial_s u(s)\|^2_{H^\kappa} + \|\nabla u(s)\|^2_{H^\kappa})ds\right) \\
&\leq C\|\boldsymbol{\omega}\|^2_{\mathcal{H}^\kappa},
\end{aligned}$$

since the $\kappa-$energy

$$E^\kappa(t,u) = \frac{1}{2}(\|\partial_t u\|^2_{H^\kappa} + \|\nabla u\|^2_{H^\kappa})$$

is also decreasing in t. Coming back to (3.8), we see that $\|g_1\|_{L^2(\mathbb{R}_+;H^\kappa)} \leq C\|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}$ and then,

$$\|\,\mathrm{e}^{-t(\Pr(a^u)+\varepsilon)}\, v(t,x)\|_{L^2(\mathbb{R}_+,H^1)} \leq C_\varepsilon \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}.$$

This is the exponential decay we are looking for, but in the integrated form. It is now easy to see that

$$\|u(t,\cdot)\|_{H^1} \leq C_\varepsilon\, \mathrm{e}^{t(\Pr(a^u)+\varepsilon)}\, \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}\,.$$

We have to check that the same property is valid for $\partial_t u$. Using the same methods as above, we also have

$$P(\tau)\mathcal{F}_{t\to-\tau}(\partial_t v) = -\tau\check{g}_1(\tau),$$

and then, $\mathcal{F}_{t\to-\tau}(\partial_t v) = -\tau R(\tau)\check{g}_1(\tau)$. It follows that

$$\begin{aligned}
\| \mathrm{e}^{-t\,\mathrm{Pr}(a^u)^+}\,\partial_t v(t,x)\|_{L^2(\mathbb{R}_+,H^0)} &= \|\check{v}(\tau+\mathrm{i}\,\mathrm{Pr}(a^u)^+\|_{L^2(\mathbb{R},H^0)} \\
&= \|\tau R(\tau+\mathrm{i}\,\mathrm{Pr}(a^u)^+)\check{g}_1(\tau+\mathrm{i}\,\mathrm{Pr}(a^u)^+,x)\|_{L^2(\mathbb{R},H^0)} \\
&\leq C_\varepsilon\|\check{g}_1(\tau+\mathrm{i}\,\mathrm{Pr}(a^u)^+,x)\|_{L^2(\mathbb{R},H^\kappa)} \\
&\leq C_\varepsilon\|g_1(t,x)\|_{L^2(\mathbb{R}_+,H^\kappa)}\,.
\end{aligned}$$

Grouping the results, we see that

$$\|\mathbf{u}\|_{\mathcal{H}} \leq C_\varepsilon\,\mathrm{e}^{t(\mathrm{Pr}(a^u)+2\varepsilon)}\,\|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}$$

and this concludes the proof of Theorem 2.

3.3. **Energy decay.** We end this section with the proof of the Theorem 3, which gives the exponential energy decay. This is an immediate consequence of the following lemma, that tells us that the energy can be controlled by the H^1 norm of u, for $t \geq 2$:

Lemma 9. *There exists $C > 0$ such that for any solution u of* (1.1) *and $E(u,t)$ the associated energy functional, we have*

$$E(u,T) \leq C\|u\|^2_{L^2([T-2,T+1];H^1)}\,,\quad T \geq 2.$$

Proof. This is a standard result, we borrow the proof from [EvZw]. For $T > 2$, we choose $\chi_2 \in C^\infty(\mathbb{R})$, $0 \leq \chi_2 \leq 1$ such that $\chi_2(t) = 1$ for $t \geq T$ and $\chi_2(t) = 0$ if $t \leq T-1$. Setting $u_2(t,x) = \chi_2(t)u(t,x)$, we have

$$(\partial_t^2 - \Delta + 2a\partial_t)u_2 = g_2$$

for $g_2 = \chi_2'' u + 2\chi_2'\partial_t u + 2a\chi_2' u$. Note that g_2 is compactly supported in t. Define now

$$E_2(u,t) = \frac{1}{2}\int_M (|\partial_t u_2|^2 + |\nabla u_2|^2)d\mathrm{vol}$$

and compute

$$\begin{aligned}
E_2'(u,t) &= \langle\partial_t^2 u_2, \partial_t u_2\rangle - \langle\Delta u_2, \partial_t u_2\rangle \\
&= -2\langle a\partial_t u_2, \partial_t u_2\rangle + \langle g_2, \partial_t u_2\rangle \\
&\leq C\int_M |\partial_t u_2|(|\partial_t u| + |u|)d\mathrm{vol} \\
&\leq C\left(E_2(u,t) + \int_M (|\partial_t u|^2 + |u|^2)d\mathrm{vol}\right)\,.
\end{aligned}$$

We remark that $E_2(u,T-1) = 0$ and $E_2(u,T) = E(u,T)$, so the Gronwall inequality on the interval $[T-1,T]$ gives

$$E(u,T) \leq C\left(\|\partial_t u\|^2_{L^2([T-1,T];L^2)} + \|u\|^2_{L^2([T-1,T];L^2)}\right)\,. \tag{3.9}$$

To complete the proof, we need to bound the term $\|\partial_t u\|^2_{L^2([T-1,T];L^2)}$. For this purpose, we choose $\chi_3 \in C^\infty(\mathbb{R})$, $0 \leq \chi_3 \leq 1$ such that $\chi_3(t) = 1$ for $t \in [T-1,T]$ and $\chi_3(t) = 0$ if $t \leq T-2$ and $t \geq T+1$. From (1.1), we get

$$
\begin{aligned}
0 &= \int_{T-2}^{T+1} \langle \chi_3^2 u, \partial_t^2 u - \Delta u + 2a\partial_t u\rangle dt \\
&= \int_{T-2}^{T+1} -\chi_3^2\langle \partial_t u, \partial_t u\rangle - 2\chi_3\chi_3'\langle u, \partial_t u\rangle + 2\chi_3^2\langle u, a\partial_t u\rangle + \chi_3^2\langle u, -\Delta u\rangle dt\,,
\end{aligned}
$$

whence

$$\|\partial_t u\|_{L^2([T-1,T];L^2)} \leq C\|u\|_{L^2([T-2,T+1];H^1)}.$$

Substituting this bound in (3.9) yields to the result. □

The Theorem 3 follows now from the preceding lemma. Let us denote by $u_j(t,x)$ and and $r_n(t,x)$ the first component of $p_{\tau_j}(t)\boldsymbol{\omega}$ and $\mathrm{e}^{-\,\mathrm{i}\, t\mathcal{B}}(1-\Pi)\boldsymbol{\omega}$ respectively. We learned above that

$$u(t,x) = \sum_{j=0}^{n} \mathrm{e}^{-\,\mathrm{i}\, t\tau_j}\, u_j(t,x) + r_n(t,x)$$

with $\|u_j(t,\cdot)\|_{H^1} \leq Ct^{m_j}\|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}$, and $\|r_n(t,\cdot)\|_{H^1} \leq C_\varepsilon\, \mathrm{e}^{t(\mathrm{Pr}(a^u)+2\varepsilon)}\, \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}$. Suppose first that the projection of $\boldsymbol{\omega}$ on E_0 vanishes, i.e. $\Pi_0\boldsymbol{\omega} = 0$. Then, from the preceding lemma we clearly have

$$E(u,t)^{\frac{1}{2}} \leq \sum_{j=1}^{n} \mathrm{e}^{t\,\mathrm{Im}\,\tau_j}\, C\|u_j(t,x)\|_{H^1} + C_\varepsilon\, \mathrm{e}^{t(\mathrm{Pr}(a^u)+2\varepsilon)}\, \|\boldsymbol{\omega}\|_{\mathcal{H}^\kappa}\,.$$

This shows Theorem 3 when $\Pi_0\boldsymbol{\omega} = 0$. But the general case follows easily: we can write $\tilde{u}(t,x) = u(t,x) - \Pi_0\boldsymbol{\omega}$ for which we have the expected exponential decay, and notice that $E(\tilde{u},t) = E(u,t)$ since $\Pi_0\boldsymbol{\omega}$ is constant.

4. Hyperbolic dispersion estimate

This last section is devoted to the proof of Proposition 5. Let $\varepsilon, \delta, \mathcal{V}, \mathcal{W}$ and $\mathrm{Op}_\hbar(\chi)$ be as in Section 2. We also set $N = T\log\hbar^{-1}$, $T > 0$.

4.1. **Decomposition into elementary Lagrangian states.** Recall that each set $\mathcal{W}_\beta \equiv \mathcal{W}_{b_0\dots b_{t_0-1}}$ in the cover $\mathcal{W}$ has the property

$$\Phi^k(\mathcal{W}_\beta) \subset \mathcal{V}_{b_k}\,,\ \ k \in [\![0, t_0-1]\!] \tag{4.1}$$

for some sequence $b_0, b_1, \dots, b_{t_0-1}$. We will say that a sequence sequence $\beta_1\dots\beta_N$ of sets $\mathcal{W}_{\beta_k}$ is adapted to the dynamics if the following condition is satisfied :

$$\forall k \in [1, N-1],\quad \Phi^{kt_0}(\mathcal{W}_{\beta_1}) \cap \mathcal{W}_{\beta_{k+1}} \neq \emptyset.$$

In this case, we can associate to the sequence $\{\beta_i\}$ a sequence $\gamma_1, \dots, \gamma_{Nt_0}$ of sets $\mathcal{V}_{\gamma_k} \subset \mathcal{V}$ which are visited at the times $0, \dots, Nt_0-1$ for some points of $\mathcal{W}_{\beta_1}$. We will only consider the sequences adapted to the dynamics. Indeed, it is clear from standard results on propagation of singularities that

$$\|\mathcal{U}_{\beta_N}\dots\mathcal{U}_{\beta_1}\| = \mathcal{O}(\hbar^\infty)$$

if the sequence is not adapted (see Appendix A), and in this case, Proposition 5 is obviously true.

We now decompose further each evolution of length t_0 in (2.3) by inserting additional quantum projectors. To unify the notations, we define for $j \in [\![1, Nt_0]\!]$ the following projectors and the corresponding open sets in T^*M :

$$\mathsf{P}_{\gamma_j} = \begin{cases} \Pi_{\beta_k} & \text{if } j-1 = kt_0,\ k \in \mathbb{N} \\ \tilde{\Pi}_{\gamma_j} & \text{if } j-1 \neq 0 \mod t_0 \end{cases}, \quad \mathsf{V}_{\gamma_j} = \begin{cases} \mathcal{W}_{\beta_k} & \text{if } j-1 = kt_0,\ k \in \mathbb{N} \\ \mathcal{V}_{\gamma_j} & \text{if } j-1 \neq 0 \mod t_0 \,. \end{cases} \tag{4.2}$$

We will also denote by $\mathsf{F}_\gamma \in C_0^\infty(T^*M)$ the function such that $\operatorname{supp} \mathsf{F}_\gamma \subset \mathsf{V}_\gamma$ and $\mathsf{P}_\gamma = \mathsf{F}_\gamma^w$.

Let us set up also a notation concerning the constants appearing in the various estimates we will deal with. Let $\ell, K \in \mathbb{N}$ be two parameters (independent of $\hbar$), and $e_1, e_2, e_3 > 0$ some fixed numbers. For a constant C depending on M and derivatives of χ, a, Φ^t (for t bounded) up to order $e_1\ell + e_2K + e_3$, we will write $C^{(\ell,K)}(M, \chi)$, or simply $C^{(K)}(M, \chi)$ if only one parameter is involved. If the constant C depends also on the cutoff functions F_γ and their derivatives, we will write

$$C = C^{(\ell,K)}(M, \chi, \mathcal{V}).$$

This is to recall us the dependence on the cutoff function χ supported inside $\mathcal{E}^\delta$, and the refined cover $\mathcal{V}$. We will sometimes use the notation $C^{(\ell,K)}(M, \mathcal{V})$ when no dependence on χ is assumed. Note that $\mathcal{V}$ depends implicitely on ε since its diameter was chosen such that (2.2) holds.

Using (4.1), standard propagation estimates give

$$\mathcal{U}^{t_0} \Pi_{\beta_1} = \mathcal{U} \mathsf{P}_{\gamma_{t_0}} \dots \mathcal{U} \mathsf{P}_{\gamma_1} + \mathcal{O}_{L^2 \to L^2}(\hbar^\infty) \,, \quad \mathcal{U} \equiv \mathcal{U}^1 \,,$$

and similar properties for $\mathcal{U}^{t_0} \Pi_{\beta_k}$, $k > 1$. Finally,

$$\mathcal{U}_{\beta_{Nt_0}} \dots \mathcal{U}_{\beta_1} \mathcal{U} \operatorname{Op}_\hbar(\chi) = \mathcal{U} \mathsf{P}_{\gamma_{Nt_0}} \dots \mathcal{U} \mathsf{P}_{\gamma_1} \mathcal{U} \operatorname{Op}_\hbar(\chi) + \mathcal{O}_{L^2 \to L^2}(\hbar^\infty) \,. \tag{4.3}$$

Take now $\Psi \in L^2(M)$. In order to show Proposition 5, we will write $\operatorname{Op}_\hbar(\chi)\Psi$ as a linear decomposition over some elementary Lagrangian states, and study the individual evolution of such elementary states by $\mathcal{U}^{Nt_0+1}$. This type of method comes back to [Ana1] and is the key tool to prove Proposition 5. The decomposition of $\operatorname{Op}_\hbar(\chi)\Psi$ is obtained by expliciting the action of $\operatorname{Op}_\hbar(\chi)$ in local coordinates (see Appendix A). When applying $\operatorname{Op}_\hbar(\chi)$ to Ψ using local charts labelled by ℓ, we get

$$\begin{aligned} [\operatorname{Op}_\hbar(\chi)\Psi](x) &= \sum_\ell \frac{1}{(2\pi\hbar)^d} \int \mathrm{e}^{\mathrm{i}\frac{\langle \eta, x - z_0 \rangle}{\hbar}} \chi(\frac{x+z_0}{2}, \eta) \varphi_\ell(z_0) \phi_\ell(x) \Psi(z_0) d\eta \, dz_0 \\ &= \sum_\ell \int \delta^\ell_{\chi, z_0}(x) \Psi(z_0) dz_0 \,, \end{aligned}$$

where we have defined

$$\delta^\ell_{\chi, z_0}(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^d} \int \mathrm{e}^{\mathrm{i}\frac{\langle \eta, x - z_0 \rangle}{\hbar}} \chi(\frac{x+z_0}{2}, \eta) \varphi_\ell(z_0) \phi_\ell(x) d\eta \,.$$

This is a Lagrangian state, which Lagrangian manifold is given by

$$\Lambda^0 \stackrel{\text{def}}{=} T^*_{z_0} M \cap \mathcal{E}^\delta \subset T^*M \,.$$

Geometrically, Λ^0 corresponds to a small, connected piece taken out of the union of spheres $\{T^*_{z_0} M \cap p^{-1}(\frac{1}{2} + \nu) \,, \ |\nu| \leq \delta\}$. If we project and evolve Ψ according to the operator appearing in the right hand side of (4.3), we get :

$$\begin{aligned}\|\mathcal{U}^t \mathsf{P}_{\gamma_{Nt_0}} \dots \mathcal{U}\mathsf{P}_{\gamma_1}\mathcal{U}\,\mathrm{Op}_\hbar(\chi)\Psi\| &\leq \sum_\ell \sup_z \|\mathcal{U}^t \mathsf{P}_{\gamma_{Nt_0}} \dots \mathsf{P}_{\gamma_1}\mathcal{U}\delta^\ell_{\chi,z_0}\| \int_M |\Psi(x)|dx \\ &\leq C\sum_\ell \sup_z \|\mathcal{U}^t \mathsf{P}_{\gamma_{Nt_0}} \dots \mathsf{P}_{\gamma_1}\mathcal{U}\delta^\ell_{\chi,z_0}\|\|\Psi\| \end{aligned} \tag{4.4}$$

where $C > 0$ depends only on the manifold M. Hence we are lead by this superposition principle to study in detail states of the form $\mathcal{U}^t \mathsf{P}_{\gamma_n} \dots \mathcal{U}\mathsf{P}_{\gamma_1}\mathcal{U}\delta^\ell_{\chi,z_0}$, for $n \in [\![1, Nt_0]\!]$ and $t \in [0,1]$. For simplicity, because the local charts will not play any role in the following, we will omit them in the formulæ.

4.2. **Evolution of Lagrangian states and their Lagrangian manifolds.**

4.2.1. *Ansatz for short times.* In this section we investigate the first step of the sequence of projection–evolution given in (4.3): our goal is to describe the state $\mathcal{U}^t\delta_{\chi,z_0}$ with $t \in [0,1]$. Since $\mathcal{U}^t$ is a Fourier integral operator, we know that $\mathcal{U}^t\delta_{\chi,z_0}$ is a Lagrangian state, supported on the Lagrangian manifold

$$\Lambda^0(t) \stackrel{\text{def}}{=} \Phi^t(\Lambda^0)\,,\ t \in [0,1].$$

Because of our assumptions on the injectivity radius, the flow $\Phi^t : \Lambda^0(s) \to \Lambda^0(t)$ for $1 \geq t \geq s > 0$, induces on M a bijection from $\pi\Lambda^0(s)$ to $\pi\Lambda^0(t)$. In other words, $\Lambda^0(t)$ projects diffeomorphically on M for $t \in]0,1]$, i.e. $\ker d\pi|_{\Lambda^0(t)} = 0$: in this case, we will say that $\Lambda^0(t)$ is *projectible*. This is the reason for introducing a first step of propagation during a time 1 : the Lagrangian manifold $\Lambda^0(0)$ is not projectible, but as soon as $t \in]0,1]$, $\Lambda^0(t)$ projects diffeomorphically. Treating separately this evolution for times $t \in [0,1]$ avoid some unnecessary technical complications.

The remark above implies that the Lagrangian manifold $\Lambda^0(t),\ t \in]0,1]$ is generated by the graph of the differential of a smooth, well defined function S_0 :

$$\Lambda^0(t) = \{(x, d_x S_0(t,x,z_0)) \,:\, 1 \geq t > 0,\ x \in \pi\Phi^t(\Lambda^0)\}\,.$$

This means that for $t \in]0,1]$, we have the Lagrangian Ansatz :

$$\begin{aligned} v^0(t,x,z_0) &\stackrel{\text{def}}{=} \mathcal{U}^t\delta_{\chi,z_0}(x) \\ &= \frac{1}{(2\pi\hbar)^{\frac{d}{2}}}\left(\mathrm{e}^{\mathrm{i}\,\frac{S_0(t,x,z_0)}{\hbar}}\sum_{k=0}^{K-1}\hbar^k b^0_k(t,x,z_0) + \hbar^K B^0_K(t,x,z_0)\right). \end{aligned} \tag{4.5}$$

The functions $b^0_k(t,x,z_0)$ are smooth, and $x \in \pi\Lambda^0(t)$. Furthermore, given any multi index ℓ, they satisfy

$$\|\partial^\ell_x b^0_k(t,\cdot,z_0)\| \leq C_{\ell,k} \tag{4.6}$$

where the constants $C_{\ell,k}$ depends only on M (via the Hamiltonian flow of p), the damping a, the cutoff function χ and their derivatives up to order $2k+\ell$. However, note that $C_{0,0}$ only depends on M. The remainder satisfies $\|B^0_K\| \leq C_K$ where the constant C_K also depends on M, a, χ and is uniformly bounded with respect to x, z_0. The base point z_0 will be fixed until section 4.5, so it will be ommited in the following to simplify the notations.

4.2.2. *Further evolution.* In the sequence of projection–evolution (4.3), we then have performed the first step, and obtained an Ansatz for $\mathcal{U}^t\delta_\chi,\ t\in]0,1]$ up to terms of order $\hbar^{K-d/2}$, for any $K\geq 0$. The main goal of the next paragraphs consist in finding an Ansatz for the full state

$$v^n(t,x)\stackrel{\rm def}{=}\mathcal{U}^t\mathsf{P}_{\gamma_n}\mathcal{U}\mathsf{P}_{\gamma_{n-1}}\ldots\mathcal{U}\mathsf{P}_{\gamma_1}\mathcal{U}\delta_\chi\,,\ t\in[0,1],\ n\geq 1\,. \tag{4.7}$$

The β_j are defined according to $j-1\mod t_0$ as in the preceding section, but here n is arbitrary in the interval $[\![1,Nt_0]\!]$. Because the operator $\mathcal{U}^t\mathsf{P}$ is a Fourier integral operator, $v^j(t,x),\ j\geq 1$ is a Lagrangian state, with a Lagrangian manifold which will be denoted by $\Lambda^j(t)$. This manifold consist in a small piece of $\Phi^{j+t}(\Lambda^0)$, because of the successive applications of the projectors P_γ between the evolution operator $\mathcal{U}$. If $j=1$, the Lagrangian manifold $\Lambda^1(0)$ is given by

$$\Lambda^1(0)=\Lambda^0(1)\cap\mathsf{V}_{\gamma_1},$$

and for $t\in[0,1]$ we have $\Lambda^1(t)=\Phi^t(\Lambda^1(0))$. For $j\geq 1$, $\Lambda^j(t)$ can be obtained by a similar procedure: knowing $\Lambda^{j-1}(1)$, we take for $\Lambda^j(t),\ t\in[0,1]$ the Lagrangian manifold

$$\Lambda^j(0)\stackrel{\rm def}{=}\Lambda^{j-1}(1)\cap\mathsf{V}_{\gamma_j}\,,\quad\text{and}\quad\Lambda^j(t)=\Phi^t(\Lambda^j(0))\,.$$

Of course, if the intersection $\Lambda^{j-1}(1)\cap\mathsf{V}_{\gamma_j}$ is empty, the construction has to be stopped, since by standard propagation estimates, v^j will be of order $\mathcal{O}(\hbar^\infty)$. But this situation will not happen since the sequence $\{\beta_i\}$ is adapted to the dynamics. It follows that

$$\forall j\in[\![1,n]\!],\quad\Lambda^j(0)\neq\emptyset\,.$$

One can show (see [AnNo], Section 3.4.1 for an argument) that the Lagrangian manifolds $\Lambda^j(t)$ are projectible for all $j\geq 1$. This is mainly because M has no conjugate points. In particular, any $\Lambda^j(t)$ can be parametrized as a graph on M of a differential, which means that there is a generating function $S_j(t,x)$ such that

$$\Lambda^j(t)=\{x,d_xS_j(t,x)\}\,.$$

By extension, we will call a Lagrangian state projectible if its Lagrangian manifold is.

Let us introduce now some notations that will be often used later. Suppose that $x\in\pi\Lambda^j(t)$, $j\geq 1$. Then, there is a unique $y=y(x)\in\pi\Lambda^j(0)$ such that

$$\pi\circ\Phi^t(y,d_yS_j(0,y))=x\,.$$

If we denote for $t\in[0,s]$ the (inverse) induced flow on M by

$$\phi^{-t}_{S_j(s)}:x\in\pi\Lambda^j(s)\mapsto\pi\Phi^{-t}\left(x,d_xS_j(s,x)\right)\in\pi\Lambda^j(s-t),$$

we have $y(x)=\phi^{-t}_{S_j(t)}(x)$. If $x\in\pi\Lambda^j(t)$, then by construction

$$\Phi^{-t-k}(x,d_xS_j(t,x))\in\Lambda^{j-k}(0)\subset\Lambda^{j-k-1}(1)\,,\ k\in[\![0,j-1]\!]\,.$$

By definition, we will write

$$\phi^{-t-k}_{S_j(t)}(x)=\pi\Phi^{-t-k}(x,d_xS_j(t,x))\quad\text{and}\quad\phi^{-k}_{S_j}(x)=\pi\Phi^{-k}(x,d_xS_j(1,x))\,.$$

To summarize, our sequence of projections and evolutions can be cast into the following way:

$$\begin{array}{ccccccccccccc}\delta_\chi & \xrightarrow{\mathcal{U}^1} & v^0(1,\cdot) & \xrightarrow{\mathsf{P}_1} & v^1(0,\cdot) & \xrightarrow{\mathcal{U}} & v^1(1,\cdot) & \xrightarrow{\mathsf{P}_2} & \cdots & \xrightarrow{\mathsf{P}_n} & v^n(0,\cdot) & \xrightarrow{\mathcal{U}^t} & v^n(t,\cdot)\\ \\ \Lambda^0 & \xrightarrow{\Phi^1} & \Lambda^0(1) & \xrightarrow{|_{\mathsf{V}_1}} & \Lambda^1(0) & \xrightarrow{\Phi^1} & \Lambda^1(1) & \xrightarrow{|_{\mathsf{V}_2}} & \cdots & \xrightarrow{|_{\mathsf{V}_n}} & \Lambda^n(0) & \xrightarrow{\Phi^t} & \Lambda^n(t)\end{array} \tag{4.8}$$

On the top line are written the successive evolutions of the Lagrangian states, while the evolution of their respective Lagrangian manifolds is written below (the notation $|_{\mathsf{V}}$ denotes a restriction to the set $\mathsf{V} \subset T^*M$).

4.3. **Evolution of a projectible Lagrangian state.** Let V_γ and P_γ be as in (4.2). The next proposition contains an explicit description of the action of the Fourier integral operators $\mathcal{U}^t\mathsf{P}$ on projectible Lagrangian states localized inside V_γ.

Proposition 10. *Let V_γ and $\mathsf{P}_\gamma = \mathsf{F}_\gamma^w$ be as in Section 4.1. Let $w_\hbar(x) = w(x)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\psi(x)}$ be a projectible Lagrangian state, supported on a projectible Lagrangian manifold*

$$\Lambda = \{x, d_x\psi(x)\} \subset \mathsf{V}_\gamma \,.$$

Assume also that $\Lambda(t) \stackrel{\text{def}}{=} \Phi^t\Lambda$ is projectible for $t \in [0,1]$. We have the following asymptotic developement :

$$[\mathcal{U}^t\mathsf{P}_\gamma w_\hbar](x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\psi(t,x)} \sum_{k=0}^{K-1} \hbar^k w_k(t,x) + \hbar^K r_K(t,x) \tag{4.9}$$

where $\psi(t,\cdot)$ is a generating function for $\Lambda(t)$. The amplitudes w_k can be computed from the geodesic flow (via the function φ_γ), the damping q and the function F_γ. Moreover, the following bounds hold :

$$\begin{aligned} \|w_k\|_{C^\ell} &\leq C_{\ell,k}\|w\|_{C^{\ell+2k}} \\ \|r_K\|_{C^\ell} &\leq C_{\ell,K}\|w\|_{C^{\ell+2K+d}} \end{aligned}$$

where the constants depend on φ_γ, a, F_γ and their derivatives up to order $\ell+2K+d$, namely $C_{\ell,k} = C^{(\ell,k)}(M,\mathcal{V})$. An explicit expression for w_k will be given in the proof.

Proof. The steps we will encounter below are very standard in the non-damping case, i.e. $q = 0$. If the diameter of the partition $\mathcal{V}$ of $\mathcal{E}^\delta$ is chosen small enough, we can assume without loss of generality the existence of a function $\varphi_\gamma \in C^\infty([0,1]\times\mathbb{R}^d\times\mathbb{R}^d)$ which generates the canonical transformation given by the geodesic flow on V_γ for times $t \in [0,1]$, in other words :

$$\forall (y,\eta) \in \mathsf{V}_\gamma\,, \quad \Phi^t(y,\eta) = (x,\xi) \Leftrightarrow \xi = \partial_x\varphi_\gamma(t,x,\eta) \text{ and } y = \partial_\eta\varphi_\gamma(t,x,\eta)\,. \tag{4.10}$$

Furthermore, φ_γ satisfies $\det \partial^2_{x,\eta}\varphi_\gamma \neq 0$, and solves the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t\varphi_\gamma + p(x, d_x\varphi_\gamma) = 0 \\ \varphi_\gamma(0,x,\eta) = \langle \eta, x\rangle\,. \end{cases}$$

We first look for an oscillatory integral representation:

$$\begin{aligned} \mathcal{U}^t\mathsf{P}_\gamma w_\hbar(x) &= \frac{1}{(2\pi\hbar)^d}\iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\varphi_\gamma(t,x,\eta)-\langle y,\eta\rangle+\psi(y))} \sum_{k=0}^{K-1}\hbar^k a_k^\gamma(t,x,y,\eta)w(y)dyd\eta \\ &\quad +\mathcal{O}_{L^2}(\hbar^K) \\ &\stackrel{\text{def}}{=} b_\hbar(t,x) + \hbar^K\tilde r_K(t,x)\,, \quad \|\tilde r_K\| = \mathcal{O}(1), \end{aligned} \tag{4.11}$$

with $(y,\eta) \in \mathsf{V}_\gamma$. For simplicity, we will omit the dependence on γ in the formulæ. We have to determine the amplitudes a_k. For this, we want $b_\hbar$ to solve

$$\frac{\partial b_\hbar}{\partial t} = (\frac{\mathrm{i}\,\hbar\Delta_g}{2} - q)b_\hbar$$

up to order $\hbar^K$. Direct computations using (1.2) show that the functions φ and a_k must satisfy the following equations :

$$\begin{cases} \partial_t\varphi + p(x, d_x\varphi) = 0 & \text{(Hamilton-Jacobi equation)} \\ \partial_t a_0 + q a_0 + X[a_0] + \frac{1}{2} a_0 \operatorname{div}_g X = 0 & \text{(0-th transport equation)} \\ \partial_t a_k + q a_k + X[a_k] + \frac{1}{2} a_k \operatorname{div}_g X = \frac{\mathrm{i}}{2}\Delta_g a_{k-1} & (k-\text{th transport equation}) \end{cases} \tag{4.12}$$

with initial conditions

$$\begin{cases} \varphi(0, x, \eta) = \langle x, \eta\rangle \\ a_0(0, x, y, \eta) = \mathsf{F}(\frac{x+y}{2}, \eta) \\ a_k(0, x, y, \eta) = 0 \text{ for } k \geq 1. \end{cases}$$

The variables y and η are fixed in these equations, so they will play the role of parameters for the moment and will sometimes be skipped in the formulæ. X is a vector field on M depending on t, and $\operatorname{div}_g X$ its Riemannian divergence. In local coordinates,

$$X = g^{ij}(x)\partial_{x_j}\varphi(t, x)\,\partial_{x_i} = \partial_{\xi_i} p(x, \partial_x\varphi(t, x))\partial_{x_i} \quad \text{and} \quad \operatorname{div}_g X = \frac{1}{\sqrt{\bar{g}}}\partial_i(\sqrt{\bar{g}} X^i).$$

The Hamilton-Jacobi equation is satisfied by construction. To deal with the transport equations, we notice that X corresponds to the projection on M of the Hamiltonian vector field H_p at $(x, d_x\varphi(t, x, \eta)) \in T^*M$. Let us call first

$$\Lambda_{t,\eta} = \{(x, d_x\varphi(t, x, \eta)), x \in \pi\Phi^t\Lambda\}, \quad \eta \text{ fixed.}$$

This Lagrangian manifold is the image of the Lagrangian manifold $\Lambda_{0,\eta} = \{(y, \eta) : y \in \pi\Lambda\}$ by the geodesic flow Φ^t. The flow κ_s^t on M generated by X can be now identified with the geodesic flow restricted to $\Lambda_{s,\eta}$:

$$\kappa_s^t : \pi\Lambda_{s,\eta} \ni x \mapsto \pi\Phi^t(x, \partial_x\varphi(t, x, \eta)) \in \pi\Lambda_{t+s,\eta}.$$

The inverse flow $(\kappa_s^t)^{-1}$ will be denoted by κ_{s+t}^{-t}. Let us extend now the flow κ_s^t of X on M to the flow $\mathcal{K}^t$ generated by the vector field $\mathcal{X} = \partial_t + X$ on $\mathbb{R} \times M$:

$$\mathcal{K}^t : \begin{cases} \mathbb{R} \times M \to \mathbb{R} \times M \\ (s, x) \mapsto (s + t, \kappa_s^t(x)) \,. \end{cases}$$

We then identify the functions a_k with Riemannian half-densities on $\mathbb{R}\times M$ – see [Dui, EvZw]:

$$a_k(t, x) \equiv a_k(t, x)\sqrt{dt d\mathrm{vol}(x)} = a_k(t, x)\sqrt{\bar{g}(x)}^{\frac{1}{2}}|dtdx|^{\frac{1}{2}} \in C^\infty(\mathbb{R} \times M, \Omega_{\frac{1}{2}}).$$

Since we have

$$\mathcal{L}_{\mathcal{X}}(a_k\sqrt{dt d\mathrm{vol}}) = (\mathcal{X}[a_k] + \frac{1}{2} a_k \operatorname{div}_g X)\sqrt{dt d\mathrm{vol}}\,,$$

the 0–th transport equation takes the simple form of an ordinary differential equation:

$$\mathcal{L}_{\mathcal{X}}(a_0\sqrt{dt d\mathrm{vol}}) + q a_0\sqrt{dt d\mathrm{vol}} = 0.$$

This is the same as

$$\frac{d}{dt}(\mathcal{K}^t)^* a_0\sqrt{dt d\mathrm{vol}} = -q a_0\sqrt{dt d\mathrm{vol}},$$

which is solved by

$$a_0\sqrt{dt d\mathrm{vol}} = \mathrm{e}^{-\int_0^t q\circ\mathcal{K}^{s-t} ds}(\mathcal{K}^{-t})^* a_0\sqrt{dt d\mathrm{vol}}\,.$$

We now have to make explicit the coordinates dependence, which yields to

$$a_0(t, x)\sqrt{\bar{g}(x)}^{\frac{1}{2}}|dxdt|^{\frac{1}{2}} = \mathrm{e}^{-\int_0^t q\circ\kappa_t^{s-t}(x) ds}\, a_0(0, \kappa_t^{-t}(x))\sqrt{\bar{g}(\kappa_t^{-t}(x))}|\det d_x\kappa_t^{-t}|^{\frac{1}{2}}|dxdt|^{\frac{1}{2}}.$$

Consequently,

$$a_0(t,x) = \mathrm{e}^{-\int_0^t q\circ\kappa_t^{s-t}(x)ds}\, a_0(0,\kappa_t^{-t}(x))\frac{\sqrt{\bar g(\kappa_t^{-t}(x))}}{\sqrt{\bar g(x)}}|\det d_x\kappa_t^{-t}|^{\frac12}\,.$$

Since

$$\kappa_t^{-t} : x \mapsto \pi\Phi^{-t}(x,\partial_x\varphi(t,x,\eta)) = \partial_\eta\varphi(t,x,\eta),$$

it is clear that $|\det d_x\kappa_t^{-t}(x)| = |\det \partial^2_{x\eta}\varphi(t,x,\eta)|$. For convenience, we introduce the following operator $\mathcal{T}_s^t$ transporting functions f on M with support inside $\pi\Lambda_{s,\eta}$ to functions on $\pi\Lambda_{t+s,\eta}$ while damping them along the trajectory :

$$\mathcal{T}_s^t(f)(x) = \mathrm{e}^{-\int_0^t q\circ\kappa_{t+s}^{\sigma-t}d\sigma}\, f(\kappa_{t+s}^{-t}(x))\frac{\sqrt{\bar g(\kappa_{t+s}^{-t}(x))}}{\sqrt{\bar g(x)}}|\det d_x\kappa_{t+s}^{-t}(x)|^{\frac12}.$$

This operator plays a crucial role, since we have

$$a_0(t,\cdot) = \mathcal{T}_0^t(a_0(0,\cdot)) = \mathcal{T}_0^t\mathsf{F}, \tag{4.13}$$

from which we see that $a_0(t,\cdot)$ is supported inside $\pi\Lambda_{t,\eta}$. By the Duhamel formula, the higher order terms can now be computed, they are given by

$$a_k(t,\cdot) = \int_0^t \mathcal{T}_s^{t-s}\left(\frac{\mathrm{i}}{2}\Delta_g a_{k-1}(s)\right)\, ds\,.$$

The ansatz $b_\hbar(t,x)$ constructed so far satisfies the approximate equation

$$\frac{\partial b_\hbar}{\partial t} = (\mathrm{i}\,\hbar\Delta_g - q)b_\hbar - \frac{\mathrm{i}}{2}\hbar^K\iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t,x,\eta,y)}\, w(y)\Delta_g a_{K-1}(t,x,y,\eta)\, dyd\eta\,.$$

The difference with the actual solution $\mathcal{U}^t\mathsf{P}$ is bounded by

$$\hbar^K t\|\Delta_g a_{K-1}\| \le Ct\hbar^K\,,$$

where $C = C^{(2K)}(M,\mathcal{V})$, so (4.11) is satisfied.

As noticed above, for time $t>0$, the state $\mathcal{U}^t\mathsf{P}w_\hbar$ is a Lagrangian state, supported on the Lagrangian manifold $\Lambda(t) = \Phi^t\Lambda$. By hypothese, $\Lambda(t)$ is projectible, so we expect an asymptotic expansion for $b_\hbar(t,x)$, exactly as in (4.5). To this end, we now proceed to the stationary phase developement of the oscillatory integral in (4.11). We set

$$I_k(x) = \frac{1}{(2\pi\hbar)^d}\iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\varphi(t,x,\eta)-\langle y,\eta\rangle+\psi(y))}\, a_k(t,x,y,\eta)w(y)dyd\eta\,.$$

The stationary points of the phase are given by

$$\begin{cases}\psi'(y) = \eta\\ \partial_\eta\varphi(t,x,\eta) = y,\end{cases}$$

for which there exists a solution $(y_c,\eta_c)\in\Lambda(0)$ in view of (4.10). Moreover, this solution is unique since $\Lambda(t)$ is projectible: $y_c = y_c(x)\in\pi\Lambda(0)$ is the unique point in $\pi\Lambda(0)$ such that $x = \pi\Phi^t(y_c,\psi'(y_c))$, and then $\eta_c = \psi'(y_c)$ is the unique vector allowing the point y_c to reach x in time t. The generating function for $\Lambda(t)$ we are looking for is then given by

$$\psi(t,x) = S(t,x,y_{c,}(x),\eta_c(x)).$$

Applying now the stationary phase theorem for each I_k (see for instance [Hör], Theorem 7.7.6, or [NoZw], Lemma 4.1 for a similar computation), summing up the results and ordering the different terms according to their associated power of $\hbar$, we see that (4.14) holds with

$$w_0(t,x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\beta(t)} \frac{a_0(t,x,y_c,\eta_c)}{|\det(1-\partial^2_{\eta\eta}\varphi(t,x,\eta_c)\circ\psi''(y_c))|^{\frac{1}{2}}} w(y_c)\,, \qquad \beta\in C^\infty(\mathbb{R}),$$

and

$$w_k(t,x) = \sum_{i=0}^{k} A_{2i}(x,D_{x,\eta})(a_{k-i}(t,x,y,\eta)w(y))|_{(y,\eta)=(y_c,\eta_c)}\,. \tag{4.14}$$

A_{2i} denotes a differential operator of order $2i$, with coefficients depending smoothly on φ, ψ and their derivatives up to order $2i+2$. This yields to the following bounds :

$$\|w_k\|_{C^\ell} \quad\leq\quad C_{\ell,k}\|w\|_{C^{\ell+2k}}$$

where $C_{\ell,k} = C^{(\ell,k)}(M,\mathcal{V})$. The remainder terms $r_K(t,x)$ is the sum of the remainders coming from the stationary phase developement of I_k up to order $K-k$. Each remainder of order $K-k$ has a C^ℓ norm bounded by $C_{\ell,K-k}\hbar^{K-k}\|w\|_{C^{\ell+2(K-k)+d}}$ so we see that

$$\|r_K\|_{C^\ell} \leq C_{\ell,K}\|w\|_{C^{\ell+2K+d}}, \quad C = C^{(\ell,K)}(M,\mathcal{V}).$$

The principal symbol w_0 can also be interpreted more geometrically. As in Section 4.2, denote by $\phi^{-t}_{\psi(t)}$ the following map

$$\phi^{-t}_{\psi(t)} : \begin{cases} \pi\Lambda(t) \to \pi\Lambda(0) \\ x \mapsto \pi\Phi^{-t}(x, d_x\psi(t,x))\,. \end{cases}$$

Let us write the differential of $\Phi^t : (y,\eta)\mapsto(x,\xi)$ as $d\Phi^t(\delta y,\delta\eta) = (\delta x,\delta\xi)$. Using (4.10), we have

$$\begin{aligned} \delta y &= \partial^2_{x\eta}\varphi\delta x + \partial^2_{\eta\eta}\varphi\delta\eta \\ \delta\xi &= \partial^2_{xx}\varphi\delta x + \partial^2_{x\eta}\varphi\delta\eta\,, \end{aligned}$$

and then, since $\partial^2_{x\eta}\varphi$ is invertible,

$$\begin{pmatrix} \delta x \\ \delta\xi \end{pmatrix} = \begin{pmatrix} \partial^2_{x\eta}\varphi^{-1} & -\partial^2_{x\eta}\varphi^{-1}\partial^2_{\eta\eta}\varphi \\ \partial^2_{xx}\varphi\partial^2_{x\eta}\varphi^{-1} & \partial^2_{x\eta}\varphi - \partial_{xx}\varphi\partial^2_{x\eta}\varphi^{-1}\partial^2_{\eta\eta}\varphi \end{pmatrix} \begin{pmatrix} \delta y \\ \delta\eta \end{pmatrix}$$

If we restrict Φ^t to $\Lambda(0)$, we have $\delta\eta = \psi''(y)\delta y$, which means that for $x\in\pi\Lambda(t)$,

$$d\phi^{-t}_{\psi(t)}(x) = \partial^2_{x\eta}\varphi(t,x,\eta_c)(1-\partial^2_{\eta\eta}\varphi(t,x,\eta_c)\psi''(y_c))^{-1}.$$

It follows from (4.14) that

$$\begin{aligned} w_0(t,x) &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\beta(t)}\, w(y_c)\mathsf{F}(y_c,\eta_c)\ \mathrm{e}^{-\int_0^1 q(\phi^{-t+s}_{\psi(t)}(x))ds}\, |\det d\phi^{-t}_{\psi(t)}(x)|^{\frac{1}{2}} \sqrt{\frac{\bar g(\phi^{-t}_{\psi(t)}(x))}{\bar g(x)}}^{\frac{1}{2}} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\beta(t)}\, w(y_c)\mathsf{F}(y_c,\eta_c)\ \mathrm{e}^{-\int_0^1 q(\phi^{-t+s}_{\psi(t)}(x))ds} \left|\mathrm{Jac}(d\phi^{-t}_{\psi(t)}(x))\right|^{\frac{1}{2}}, \end{aligned}$$

where $\mathrm{Jac}(f)$ denotes the Jacobian of $f: M\to M$ measured with respect to the Riemannian volume. □

4.4. **Ansatz for $n > 1$.** In this paragraph, we construct by induction on n a Lagrangian state $b^n(t,x)$ supported on $\Lambda^n(t)$, in order to approximate $v^n(t,x)$ up to order $\hbar^{K-d/2}$.

Proposition 11. *There exists a sequence of functions*

$$\{b_k^n(t,x), S_n(t,x) \ : \ n \geq 1,\ k < K,\ x \in M,\ t \in [0,1]\}$$

such that $S_n(t,x)$ is a generating function for $\Lambda^n(t)$, and

$$v^n(t,x) = \frac{1}{(2\pi\hbar)^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\frac{S_n(t,x)}{\hbar}} \sum_{k=0}^{K-1} \hbar^k b_k^n(t,x) + \hbar^{K-\frac{d}{2}} R_K^n(t,x) \tag{4.15}$$

where R_K^n satisfies

$$\|R_K^n\| \leq C_K(1+C\hbar)^n \left(\sum_{i=2}^{n} \sum_{k=0}^{K-1} \|b_k^{i-1}(1,\cdot)\|_{C^{2(K-k)+d}} + C' \right) \tag{4.16}$$

where $C' = C^{(K)}(M,\chi)$, $C_K = C^{(K)}(M,\chi,\mathcal{V})$ and $C > 0$ is fixed.

Proof. The construction of the amplitudes b_k^n for all $k \geq 0$ is done by induction on n, following step by step the sequence (4.8). In Section 4.2 we obtained $\mathcal{U}^1\delta_\chi$ as a projectible Lagrangian state:

$$\begin{aligned} v^0(1,x) &= \frac{1}{(2\pi\hbar)^{d/2}} \mathrm{e}^{\mathrm{i}\frac{S_0(1,x)}{\hbar}} \sum_{k=0}^{K-1} \hbar^k b_k^0(1,x) + \hbar^{K-d/2} B_K^0(1,x) \\ &\stackrel{\text{def}}{=} \frac{1}{(2\pi\hbar)^{d/2}} b^0(1,x) + \hbar^{K-d/2} R_K^0(1,x)\,, \end{aligned}$$

and we know that $b^0(1,\cdot)$ satisfies the hypotheses of Proposition 10, which will be used to describe $\mathcal{U}^t\mathsf{P}_{\gamma_1}v^0(1,\cdot)$. More generally, suppose that the preceding step has lead for some $n \geq 1$ to

$$\begin{aligned} v^{n-1}(t,x) &= \frac{1}{(2\pi\hbar)^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\frac{S_{n-1}(t,x)}{\hbar}} \sum_{k=0}^{K-1} \hbar^k b_k^{n-1}(t,x) + \hbar^{K-d/2} R_K^{n-1}(t,x) \\ &= \frac{1}{(2\pi\hbar)^{\frac{d}{2}}} b^{n-1}(t,x) + \hbar^{K-d/2} R_K^{n-1}(t,x) \end{aligned}$$

where $b^{n-1}(t,\cdot)$ is a Lagrangian state, supported on the Lagrangian manifold $\Lambda^{n-1}(t)$, and R_K^{n-1} is some remainder in $L^2(M)$. We now apply Proposition 10 to each Lagrangian state $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{n-1}(1,x)}\,\hbar^k b_k^{n-1}(1,x)$ appearing in the definition of b^{n-1}. Because of the term $\hbar^k$, if we want an Ansatz as in (4.15), it is enough to describe $\mathcal{U}^t\mathsf{P}_{\gamma_n}v_k^{n-1}(1,\cdot)$ up to order $K-k$, which gives a remainder of order $C_{K-k}\hbar^{K-k}\|b_k^{n-1}(1,\cdot)\|_{C^{2(K-k)+d}}$. Grouping the terms corresponding to the same power of $\hbar$ when applying Proposition 10 to each $(v_k^{n-1})_{0\leq k<K-1}$ yields to

$$[\mathcal{U}^t\mathsf{P}_{\gamma_n}b^{n-1}](x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_n(t,x)} \sum_{k=0}^{K-1} \hbar^k b_k^n(t,x) + \hbar^K B_K^n(t,x) \stackrel{\text{def}}{=} b^n(t,x) + \hbar^K B_K^n(t,x),$$

where $S_n(t,x)$ is a generating function of the Lagrangian manifold

$$\Lambda^n(t) = \Phi^t(\Lambda^{n-1}(1) \cap \mathsf{V}_{\gamma_n}).$$

The coefficients b_k^n are given by

$$b_k^n(t,x) = \sum_{i=0}^{k}\sum_{l=0}^{k-i} A_{2l}(a_{k-i-l}^{\gamma_n}(t,x,y,\eta) b_i^{n-1}(1,y))|_{(y,\eta)=(y_c,\eta_c)} \tag{4.17}$$

where $y_c = \phi_{S_n(t)}^{-t}(x)\,,\ \eta_c = d_y S_{n-1}(1,y_c)$. In particular, $b_0^n(t,x) = \mathcal{D}_n(t,x) b_0^{n-1}(1,y_c)$, with

$$\mathcal{D}_n(t,x) = \mathrm{e}^{-\int_0^1 q(\phi_{S_n(t)}^{s-t}(x))ds} \left|\mathrm{Jac}(d\phi_{S_n(t)}^{-t}(x))\right|^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\beta_n(t)}\, \mathsf{F}_{\gamma_n}(y_c,\eta_c) \tag{4.18}$$

for some $\beta_n(t)\in C^\infty(\mathbb{R})$. The remainder B_K^n satisfies

$$\|B_K^n(t)\| \le C_K \sum_{k=0}^{K-1} \|b_k^{n-1}(1,\cdot)\|_{C^{2(K-k)+d}}\,,\quad C_K = C^{(K)}(M,\chi,\mathcal{V}). \tag{4.19}$$

Hence, we end up with

$$\begin{aligned} v^n(1,x) &= \frac{1}{(2\pi\hbar)^{\frac{d}{2}}}\left(\mathrm{e}^{\mathrm{i}\,\frac{S_n(1,x)}{\hbar}} \sum_{k=0}^{K-1} \hbar^k b_k^n(1,x) + \hbar^K (B_K^n(1,x) + \mathcal{U}^1 \mathsf{P}_{\gamma_1} B_K^{n-1}(1))\right) \\ &\stackrel{\mathrm{def}}{=} \frac{1}{(2\pi\hbar)^{\frac{d}{2}}} b^n(1,x) + \hbar^{K-d/2} R_K^n(1,x) \end{aligned}$$

where $R_K^n(1,x) = (2\pi)^{-\frac{d}{2}}(B_K^n(1,x) + \mathcal{U}^1\mathsf{P}_{\gamma_n} R_K^{n-1}(1,\cdot))$. Again, b^n satisfies the hypotheses of Proposition 10, so we can continue iteratively. To complete the proof, we now have to take all the remainders into account. From the discussion above, we get :

$$\begin{aligned} \mathcal{U}^t\mathsf{P}_{\gamma_n} v^{n-1}(1,\cdot) &= (2\pi\hbar)^{-\frac{d}{2}}\mathcal{U}^t\mathsf{P}_{\gamma_n} b^{n-1}(1,\cdot) + \hbar^{K-d/2}\mathcal{U}^t\mathsf{P}_{\gamma_n}(R_K^{n-1}(1,\cdot)) \\ &= (2\pi\hbar)^{-\frac{d}{2}}\left(b^n(t,\cdot) + \hbar^K B_K^n(t.\cdot)\right) + \hbar^{K-d/2}\mathcal{U}^t\mathsf{P}_{\gamma_n}(R_K^{n-1}(1,\cdot)) \\ &= (2\pi\hbar)^{-\frac{d}{2}} b^n(t,\cdot) + \hbar^{K-d/2} R_K^n(t,\cdot) \end{aligned}$$

where we defined $R_K^n = (2\pi)^{-\frac{d}{2}} B_K^n + \mathcal{U}^t\mathsf{P}_{\gamma_n}(R_K^{n-1})$. Since $|\mathsf{F}_{\gamma_n}|\le 1$, we have

$$\|\mathcal{U}^t\mathsf{P}_{\gamma_n}\|_{L^2\to L^2} \le 1 + C\hbar,\quad C>0.$$

This implies that $\|\mathcal{U}^t\mathsf{P}_{\gamma_n}(R_K^{n-1}(1,\cdot))\| \le (1+C\hbar)\|R_K^{n-1}(1,\cdot)\|$, and finally R_K^n satisfies

$$\|R_K^n\| \le (1+C\hbar)^n\left(\|B_K^n\| + \|B_K^{n-1}\| + \ldots \|B_K^1\| + \|B_K^0\|\right) \tag{4.20}$$

In view of (4.19) and (4.5), this concludes the proof. □

Given $v^{n-1}(1,\cdot)$, we have then constructed $v^n(t,x)$ as in (4.15), but it remains to control the remainder R_K^n in L^2 norm : from (4.19) and (4.20), we see that it is crucial for this to estimate properly the C^ℓ norms of the coefficients b_k^j for $j\ge 1$ and $k\in[\![0,K-1]\!]$.

Lemma 12. *Let $n\ge 1$, and define*

$$\boldsymbol{\mathcal{D}}_n = \sup_{x\in\pi\Lambda^n(1)} \left|\prod_{i=0}^{n-1} \mathcal{D}_{n-i}(1,\phi_{S_n}^{-i}(x))\right|,\quad \boldsymbol{\mathcal{D}}_0 = 1.$$

If $x\in\pi\Lambda^n(1)$, the principal symbol b_0^n is given by

$$b_0^n(1,x) = \left(\prod_{j=0}^{n-1}\mathcal{D}_{n-j}(1,\phi_{S_n}^{-j}(x))\right) b_0^0(1,\phi_{S_n}^{-n}(x)). \tag{4.21}$$

For $k\in[\![0,K-1]\!]$, the functions b_k^n satisfy

$$\|b_k^n(1,\cdot)\|_{C^\ell} \leq C_{k,\ell}(n+1)^{3k+\ell}\boldsymbol{\mathcal{D}}_n \tag{4.22}$$

where $C_{k,\ell} = C^{(\ell,k)}(M,\chi,\mathcal{V})$. It follows that

$$\|B_K^n(1,\cdot)\| \leq C_K n^{3K+d}\boldsymbol{\mathcal{D}}_{n-1} \tag{4.23}$$

$$\|R_K^n(1)\| \leq C_K(1+C\hbar)^n \sum_{j=1}^n j^{3K+d}\boldsymbol{\mathcal{D}}_{j-1} \tag{4.24}$$

where $C>0$ and $C_K = C^{(K)}(M,\chi,\mathcal{V})$. On the other hand, if $x \notin \pi\Lambda^n(1)$, we have $b_k^n(x)=0$ for $k \in [\![0,K-1]\!]$.

Proof. First, if $x \notin \pi\Lambda^n(1)$, then there is no $\rho \in \mathsf{V}_{\gamma_n}$ such that $\pi\Phi^1(\rho) = x$, and then $v^n(1,x) = \mathcal{O}(\hbar^\infty)$. In what follows, we then consider the case $x \in \pi\Lambda^n(1)$. We first see that (4.21) simply follows from (4.18) applied recursively. If $\rho_n = (x_n,\xi_n) \in \Lambda^n(1)$, we call $\rho_j = (x_j,\xi_j) = \Phi^{j-n}(\rho_n) \in \Lambda^j(1)$ if $j \geq 0$. In other words,

$$\forall j \in [\![1,n]\!], \quad x_{j-1} = \phi_{S_j}^{-1}(x_j).$$

It will be useful to keep in mind the following sequence, which illustrates the backward trajectory of $\rho_n \in \Lambda^n(1)$ under Φ^{-k}, $k \in [\![1,n]\!]$ and its projection on M :

$$\begin{array}{ccccccccc}
\rho_0 \in \Lambda^0(1) & \xleftarrow{\Phi^{-1}} & \rho_1 \in \Lambda^1(1) & \xleftarrow{\Phi^{-1}} \cdots \xleftarrow{\Phi^{-1}} & \rho_{n-1} \in \Lambda^{n-1}(1) & \xleftarrow{\Phi^{-1}} & \rho_n \in \Lambda^n(1) \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
x_0 & \xleftarrow{\phi_{S_1}^{-1}} & x_1 & \xleftarrow{\phi_{S_2}^{-1}} \cdots \xleftarrow{\phi_{S_{n-1}}^{-1}} & x_{n-1} & \xleftarrow{\phi_{S_n}^{-1}} & x_n
\end{array}$$

We denote schematically the Jacobian matrix $d\phi_{S_j}^{-i} = \frac{\partial x_{j-i}}{\partial x_j}$ for $1 \leq i \leq j \leq n$. Since for any $E>0$, the sphere bundle $T_z^*M \cap p^{-1}(E)$ is transverse to the stable direction [Kli], the Lagrangians $\Lambda^n \subset \Phi^n\Lambda^0$ converge exponentially fast to the weak unstable foliation as $n \to \infty$. This implies that $\Phi^t|_{\Lambda^0}$ is asymptotically expanding as $t \to \infty$, except in the flow direction. Hence, the inverse flow $\Phi^{-t}|_{\Lambda^n}$ acting on Λ^n and its projection $\phi_{S_n}^{-t}$ on M have a tangent map uniformly bounded with respect to n,t. As a result, the Jacobian matrices $\partial x_{j-i}/\partial x_j$ are uniformly bounded from above : for $1 \leq i \leq j \leq n$ there exists $C = C(M)$ independent of n such that

$$\left\|\frac{\partial x_{j-i}}{\partial x_j}\right\| \leq C\,. \tag{4.25}$$

It follows that if we denote $D_j = \sup_{x_j} \mathcal{D}_j(1,x_j)$, there exists $C = C(M) > 0$ such that

$$C^{-1} \leq D_j \leq C. \tag{4.26}$$

Note also that

$$\sup_{x \in \pi\Lambda^n(1)} \left|\prod_{j=0}^{n-1} \mathcal{D}_{n-j}(1,\phi_{S_n}^{-j}(x))\right| = \prod_{j=0}^{n-1} D_{n-j} = \boldsymbol{\mathcal{D}}_n.$$

We first establish the following crucial estimate :

Lemma 13. *Let $n \geq 1$, and $k \in [\![1,n]\!]$. For every multi index α of length $|\alpha| \geq 2$, there exists a constant $C_\alpha > 0$ depending on M such that*

$$\left\|\frac{\partial^\alpha x_{n-k}}{\partial x_n^\alpha}\right\| \leq C_\alpha k^{\alpha-1} \tag{4.27}$$

Proof. We proceed by induction on k, from $k=1$ to $k=n$. The case $k=1$ is clear. Let us assume now that

$$\left\| \frac{\partial^\alpha x_{n-k'}}{\partial x_n^\alpha} \right\| \leq C_\alpha {k'}^{\alpha-1}, \quad k' \in [\![1, k-1]\!]$$

and let us show the bound for $k'=k$. For simplicity, we will denote

$$\partial_j^\alpha \stackrel{\text{def}}{=} \frac{\partial^\alpha}{\partial x_j^\alpha}, \quad \partial^\alpha x_j = \frac{\partial^\alpha x_j}{\partial x_{j+1}^\alpha}.$$

In particular, $\|\partial^\alpha x_j\| \leq C_\alpha$. We also recall the Faà di Bruno formula : let Π be the set of partitions of the ensemble $\{1, ..., |\alpha|\}$, and for $\pi \in \Pi$, write $\pi = \{B_1, ...B_k\}$ where B_i is some subset of $\{1, ..., |\alpha|\}$. Here $|\alpha| \geq k \geq 1$, and we denote $|\pi| = k$. For two smooth functions $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $f \circ g$ is well defined, one has

$$\partial^\alpha f \circ g \equiv \sum_{\pi \in \Pi} \partial^{|\pi|} f(g) \prod_{B \in \pi} \partial^B g. \tag{4.28}$$

The term in the right hand side is written schematically, to indicates a sum of derivatives of f of order $|\pi|$, times a product of $|\pi|$ terms, each of them corresponding to derivatives of g of order $|B|$. It is important for our purpose to note that $\sum |B| = |\alpha|$. Continuing from theses remarks, we compute

$$\begin{aligned} X_k \stackrel{\text{def}}{=} \partial_n^\alpha x_{n-k} &= \partial x_{n-k} \partial_n^\alpha x_{n-k+1} \\ &\quad + \sum_{\pi \in \Pi, |\pi| > 1} \partial^{|\pi|} x_{n-k} \prod_{B \in \pi} \partial_n^B x_{n-k+1} \stackrel{\text{def}}{=} \partial x_{n-k} X_{k-1} + Y_{k-1}. \end{aligned}$$

By the induction hypothesis,

$$\|Y_i\| \leq C_\alpha i^{\alpha-2} \tag{4.29}$$

since the partitions π involved in the sum contains at least two elements. Setting $M_{k-1} = \partial x_{n-k}$, we have

$$\begin{aligned} X_k &= M_{k-1} \dots M_1 X_1 + M_{k-2} \dots M_1 Y_1 + M_{k-3} \dots M_1 Y_2 \\ &\quad + \dots + M_1 Y_{k-1}. \end{aligned}$$

From the chain rule we have

$$\frac{\partial x_{j-i}}{\partial x_j} = \frac{\partial x_{j-i}}{\partial x_{j-i+1}} \dots \frac{\partial x_{j-1}}{\partial x_j},$$

and (4.25) yields to $\|M_{i-1} \dots M_1\| = \mathcal{O}(1)$ for $2 \leq i \leq k$. Adding up all the terms contributing to X_k and taking (4.29) into account yields to

$$\|X_k\| \leq C_\alpha (1 + 1^{\alpha-2} + 2^{\alpha-2} + \dots + (k-1)^{\alpha-2}) \leq C_\alpha k^{\alpha-1}$$

and the lemma is proved. □

We now prove (4.22). For this, we will proceed in two steps. First, we show the bounds for the principal symbol b_0^n. Then, we treat the higher order terms $b_k^n, k \geq 1$ using the bounds on $\|b_0^n\|_{C^\ell}$ for any ℓ. For b_0^n, The C^0 norm estimate follows directly from (4.21). From now on, we denote for convenience

$$\mathcal{D}_0(x_0) \stackrel{\text{def}}{=} b_0^0(1, x_0).$$

Computing

$$\partial_n^\ell b_0^n(x_n) = \partial_n^\ell (\mathcal{D}_n(x_n) \dots \mathcal{D}_1(x_1) \mathcal{D}_0(x_0)),$$

we will obtain a sum of terms, each of them of the form

$$M_{\alpha_n\dots\alpha_0} = \partial_n^{\alpha_n}\mathcal{D}_n\partial_n^{\alpha_{n-1}}\mathcal{D}_{n-1}\dots\partial_n^{\alpha_1}\mathcal{D}_1\partial_n^{\alpha_0}\mathcal{D}_0,$$

with $\alpha_n+\dots+\alpha_0=\ell$. Note that if ℓ is fixed with respect to n, most of the multi-indices α_i vanish when n becomes large : actually, at most $|\ell|$ are non-zero, and we will denote them by $\alpha_{i_1},\dots,\alpha_{i_k}$, $k\leq|\ell|$. Hence the above expression is made of long strings of $\mathcal{D}_i$, alternating with some derivative terms $\partial_n^{\alpha_i}\mathcal{D}_i$ which number depends only on ℓ. We can then write

$$\|M_{\alpha_n\dots\alpha_0}\|_{C^0} \leq \boldsymbol{\mathcal{D}}_n \times \frac{\|\partial_n^{\alpha_{i_1}}\mathcal{D}_{i_1}\dots\partial_n^{\alpha_{i_{k-1}}}\mathcal{D}_{i_{k-1}}\partial_n^{\alpha_{i_k}}\mathcal{D}_{i_k}\|_{C^0}}{D_{i_1}\dots D_{i_k}}\,. \tag{4.30}$$

Let us examinate each terms $\partial_n^{\alpha}\mathcal{D}_i$ appearing in the right hand side individually. By the Faà di Bruno formula and Lemma 13, we have for $i\neq 0$

$$\partial_n^{\alpha}\mathcal{D}_i(x_i) = \sum_{\pi}\partial_i^{|\pi|}\mathcal{D}_i\prod_{B\in\pi}\partial_n^{B}x_i \leq C_\pi n^{\alpha-|\pi|}\leq C_\alpha n^{\alpha-1} \tag{4.31}$$

where $C_\alpha = C^{(\ell,K)}(M,\chi,\mathcal{V})$. Of course, if $i=0$, $\|\partial_0^\alpha\mathcal{D}_0(x_0)\|_{C^0}\leq C_\alpha\|\partial_0^\alpha b_0^0\|_{C^0}$ for some constant $C_\alpha>0$. Now, for a fixed configuration of derivatives $\{\alpha\}=\{\alpha_{i_1},\dots\alpha_{i_k}\}$ we have to choose $i_1,\dots,i_k$ indices among $n+1$ to form the right hand side in (4.30), and the number of such choices is at most of order $\mathcal{O}((n+1)^k)$. Hence,

$$\begin{aligned}
\|\partial_n^\ell b_0^n\|_{C^0} &\leq \boldsymbol{\mathcal{D}}_n\sum_{\{\alpha\}}\sum_{i_i,\dots,i_k}\frac{\|\partial_n^{\alpha_{i_1}}\mathcal{D}_{i_1}\dots\partial_n^{\alpha_{i_{k-1}}}\mathcal{D}_{i_{k-1}}\partial_n^{\alpha_{i_k}}\mathcal{D}_{i_k}\|_{C^0}}{D_{i_1}\dots D_{i_k}} \\
&\leq \boldsymbol{\mathcal{D}}_n\sum_{\{\alpha\}}C_\alpha(n+1)^k(n+1)^{\alpha_1-1}\dots(n+1)^{\alpha_k-1} \\
&\leq C_\ell\boldsymbol{\mathcal{D}}_n(n+1)^\ell \qquad (4.32)
\end{aligned}$$

where $C_\ell=C^{(\ell)}(M,\chi,\mathcal{V})$. For higher order terms ($b_k^n,\ k>0$), we remark from (4.17) that we can write

$$b_k^n(x_n) = \mathcal{D}_n(x_n)b_k^{n-1}(x_{n-1}) + \sum_{j=1}^{k}\sum_{|\alpha|\leq 2j}\Gamma_{j\alpha}^n(x_n)\partial_{n-1}^\alpha b_{k-j}^{n-1}(x_{n-1})\,. \tag{4.33}$$

The function $\Gamma_{j\alpha}^n$ can be expressed with the flow, the damping and the cutoff function F_{γ_n}. It follows that the norms $\|\Gamma_{j\alpha}^n\|_{C^\ell}$ are uniformly bounded with respect to n:

$$\|\Gamma_{j,\alpha}^n\|_{C^\ell} = C^{(\ell,K)}(M,\chi,\mathcal{V}).$$

In order to show the bounds (4.22) for $k>0$, we will proceed by induction on the index k. The case $k=0$ has been treated above. Suppose now that for any ℓ and $k'\in[\![0,k-1]\!]$ we have proven

$$\|\partial_n^\ell b_{k'}^n\|_{C^0}\leq C_\ell(n+1)^{3k'+\ell}\boldsymbol{\mathcal{D}}_n,\quad C_\ell = C^{(\ell,K)}(M,\chi,\mathcal{V})\,.$$

As above, to treat the case $k'=k$, we begin by the situation where $\ell=0$. To shorten the formulæ, we introduce for $1\leq i\leq j\leq n$ the functions

$$\begin{aligned}
\boldsymbol{\Gamma}^{i,k}(x_n) &= \sum_{j=1}^{k}\sum_{|\alpha|\leq 2j}\Gamma_{j\alpha}^i(x_i)\partial_{i-1}^\alpha b_{k-j}^{i-1}(x_{i-1}) \\
\boldsymbol{J}_i^j(x_n) &= \mathcal{D}_j(x_j)\mathcal{D}_{j-1}(x_{j-1})\dots\mathcal{D}_i(x_i)
\end{aligned}$$

where the x_i, $i \leq n$ have to be considered as functions of x_n, namely $x_i = \phi_{S_n}^{-n+i}(x_n)$. Iterating (4.33) further, we have :

$$\begin{aligned} b_k^n(x_n) &= \boldsymbol{J}_n^n b_k^{n-1}(x_{n-1}) + \boldsymbol{\Gamma}^{n,k} \\ &= \boldsymbol{J}_n^n(\boldsymbol{J}_{n-1}^{n-1} b_k^{n-2}(x_{n-2}) + \boldsymbol{\Gamma}^{n-1,k}) + \boldsymbol{\Gamma}^{n,k} \\ &= \boldsymbol{J}_{n-1}^n b_k^{n-2}(x_{n-1}) + \boldsymbol{J}_n^n \boldsymbol{\Gamma}^{n-1,k} + \boldsymbol{\Gamma}^{n,k} \\ &= \boldsymbol{J}_1^n b_k^0(x_0) + \boldsymbol{J}_2^n \boldsymbol{\Gamma}^{1,k} + \boldsymbol{J}_3^n \boldsymbol{\Gamma}^{2,k} + \cdots + \boldsymbol{J}_n^n \boldsymbol{\Gamma}^{n-1,k} + \boldsymbol{\Gamma}^{n,k} \end{aligned} \tag{4.34}$$

By the induction hypothesis and (4.26), each term $\boldsymbol{\Gamma}^{i,k}$, $i > 0$ satisfies

$$\|\boldsymbol{\Gamma}^{n-i,k}\|_{C^0} \leq C_k (n-i)^{3k-1} \boldsymbol{\mathcal{D}}_{n-i}$$

hence adding up all the terms we get

$$\|b_k^n\|_{C^0} \leq C_k \boldsymbol{\mathcal{D}}_n (b_k^0(x_0) + \sum_{i=0}^{n-1} (n-i)^{3k-1}) \leq C_k \boldsymbol{\mathcal{D}}_n (n+1)^{3k}$$

and we obtain the bounds (4.22) for $\ell = 0$. To evaluate $\partial^\ell b_k^n$, $\ell > 1$, we start from the expression (4.34). We notice first that

$$\partial_n^\beta \boldsymbol{\Gamma}^{n-i,k} = \sum_{\beta_1+\beta_2=\beta} \sum_{j=1}^{k} \sum_{|\alpha|\leq 2j} (\partial_n^{\beta_1} \Gamma_{j\alpha}^{n-i}(x_{n-i}))(\partial_n^{\beta_2} \partial^\alpha b_{k-j}^{n-i-1}(x_{n-i-1}))\,.$$

Using the Faà di Bruno formula and Lemma 13, we get

$$\|\partial_n^{\beta_1} \Gamma_{j\alpha}^{n-i,k}(x_{n-i})\|_{C^0} \leq C_{\beta_1} i^{\beta_1 - 1} \quad \text{and} \quad \|\partial_n^{\beta_2} \partial^\alpha b_{k-j}^{n-i-1}(x_{n-i-1})\|_{C^0} \leq C_{\beta_2} i^{3k-1+\beta_2},$$

and this implies

$$\|\partial_n^\beta \boldsymbol{\Gamma}^{n-i,k}\|_{C^0} \leq C_\beta i^{3k-1+\beta}.$$

Then, exactly the same strategy used to derive (4.32) shows that

$$\|\partial_n^\ell \boldsymbol{J}_{i+1}^n \boldsymbol{\Gamma}^{i,k}\|_{C^0} \leq C_\ell n^{3k-1+\ell}.$$

Using these estimates and (4.34) yields to

$$\|\partial_n^\ell b_k^n\|_{C^0} \leq C_\ell (n+1) n^{3k-1+\ell} \leq C_\ell (n+1)^{3k+\ell},$$

where the constant C_ℓ is such that $C_\ell = C^{(\ell,K)}(M, \chi, \mathcal{V})$.

□

4.5. **The main estimate : proof of Proposition 5.** As noted before, the Lagrangians Λ^n converge exponentially fast as $n \to \infty$ to the weak unstable foliation. This implies that for $x \in \pi\Lambda^j(1)$, the Jacobians $J_{S_j}(x) \stackrel{\text{def}}{=} |\det \phi_{S_j(1)}^{-1}(x)|$ satisfy

$$\forall j \geq 2,\ \forall (x,\xi) \in \Lambda^j(1),\ \left| \frac{J_{S_j}(x)}{J_{S^u(x,\xi)}(x)} - 1 \right| \leq C\,\mathrm{e}^{-j/C}\,,\ C = C(M) > 0.$$

Here, S^u generates the (Lagrangian) local weak instable manifold at point (x,ξ). Moreover, theses Jacobians decay exponentially with j as $j \to \infty$. This means that uniformly with respect to n,

$$\prod_{j=0}^{n-1} J_{S_{n-j}}(\phi_{S_n}^{-j}(x)) \leq C(M) \prod_{j=0}^{n-1} J_{S^u(\Phi^{-j}(x,\xi))}(\phi_{S_n}^{-j}(x))\,.$$

The Jacobian $J_{S^u(x,\xi)}(x)$ measures the contraction of Φ^{-1} along the unstable subspace $E^u(\Phi^1(\rho))$, where $\Phi^1(\rho) = (x,\xi)$, and $x \in M$ serves as coordinates to compute this Jacobian (via the projection π). The unstable Jacobian $J^u(\rho) \stackrel{\text{def}}{=} |\det\big(d\Phi^{-1}|_{E^{u,0}(\Phi(\rho))}\big)|$ defined in

Section 2.1 express also this contraction, but in different coordinates: for n large enough, the above inequality can then be extended to

$$\prod_{j=0}^{n-1} J_{S_{n-j}}(\phi_{S_n}^{-j}(x)) \leq C \prod_{j=0}^{n-1} J_{S^u(\Phi^{-j}(x,\xi))}(\phi_{S_n}^{-j}(x)) \leq \tilde{C} \prod_{j=0}^{n-1} J^u(\Phi^{-j}(\rho)) \,. \tag{4.35}$$

where $C, \tilde{C}$ only depends on M. As noted above, because of the Anosov property of the geodesic flow, the above products decay exponentially with n. Together with the fact that the damping function is positive, it follows that the right hand side in (4.23) also decay exponentially with n. Recall now that $1 \leq n \leq Nt_0$ and $N = T \log \hbar^{-1}$. Using (4.24), we then see that the remainders R_K^n in (4.15) are uniformly bounded : they satisfy

$$\|R_K^n\| \leq C_K \,, \quad C_K = C^{(K)}(M, \chi, \mathcal{V})$$

uniformly in n and z_0, the point on which δ_{χ, z_0} was based. From the very construction of $b^n(t,x)$, we then have

$$\|\mathcal{U}^1 \mathsf{P}_{\gamma_n} \ldots \mathcal{U}^1 \mathsf{P}_{\gamma_1} \mathcal{U}^1 \delta_\chi - \frac{1}{(2\pi\hbar)^{d/2}} b^n(1,\cdot)\| \leq C_K \hbar^{K-d/2}. \tag{4.36}$$

But the bounds on the symbols b_k^n, $k > 0$ given in Lemma 12 tells us that (4.36) also holds if we replace the full symbol b^n by the principal symbol b_0^n, provided $\hbar$ is chosen small enough – say $\hbar \leq \hbar_0(\varepsilon)$. Hence, for $\hbar \leq \hbar_0$,

$$\|\mathcal{U}^1 \mathsf{P}_{\gamma_n} \ldots \mathcal{U}^1 \mathsf{P}_{\gamma_1} \mathcal{U}^1 \delta_\chi\| \quad \leq \quad (2\pi\hbar)^{-\frac{d}{2}} \|b_0^n(1,\cdot)\| + C_K \hbar^{K-d/2} \,.$$

Now, using (4.21), (4.35) and the fact that $|\mathsf{F}_\gamma| \leq 1$, we conclude that for a^u as in (1.8),

$$\|b_0^n(1,x)\| \leq C \, \mathrm{e}^{n\mathcal{O}(\hbar)} \sup_{x \in \pi\Lambda^n(1)} \exp \sum_{j=1}^{n} a^u \circ \Phi^{-j}(x, d_x S_n(1,x))$$

Here, $C = C(M)$ depends only on the manifold M. Let us consider now the particular case $n = Nt_0$ with $N = T \log \hbar^{-1}$. It follows immediately that

$$\sup_{x \in \pi\Lambda^{Nt_0}(1)} \exp \sum_{j=1}^{Nt_0} a^u \circ \Phi^{-j}(x, d_x S_n(1,x) \leq \prod_{k=1}^{N} \sup_{\rho \in \mathcal{W}_{\beta_k}} \left(\exp \sum_{j=0}^{t_0-1} a^u \circ \Phi^j(\rho) \right) .$$

By the superposition principle already mentionned in (4.4), we then obtain for some $C = C(M) > 0$ depending only on M:

$$\begin{aligned} \|\mathcal{U}\mathsf{P}_{\gamma_{Nt_0}} \ldots \mathcal{U}\mathsf{P}_{\gamma_1} \mathcal{U}^1 \operatorname{Op}_\hbar(\chi)\| \quad &\leq \quad C \sum_\ell \sup_{z_0} \|\mathcal{U}\mathsf{P}_{\gamma_{Nt_0}} \ldots \mathsf{P}_{\gamma_1} \mathcal{U}^1 \delta_{z,\alpha_0}^\ell\| \\ &\leq \quad C\hbar^{-d/2} \|b_0^n\| + C_K \hbar^{K-d/2} \\ &\leq \quad C\hbar^{-\frac{d}{2}} \prod_{k=1}^{N} \sup_{\rho \in \mathcal{W}_{\beta_k}} \left(\exp \sum_{j=0}^{t_0-1} a^u \circ \Phi^j(\rho) \right) \end{aligned}$$

To get the last line, we have noticed that K can be chosen arbitrary large: since $n \leq Tt_0 \log \hbar^{-1}$, we see that for $\hbar$ small enough, the main term in the right hand side of the second line is larger than the remainder $C_K \hbar^{K-d/2}$, and $\mathrm{e}^{Nt_0 \mathcal{O}(\hbar)} = \mathcal{O}(1)$. This completes the proof of Proposition 5.

Appendix A. Semiclassical analysis on compact manifolds

In this appendix we gather standard notions of pseudodifferential calculus on a compact, d dimensional manifold M endowed with a Riemannian structure coming from a metric g. As usual, M is equiped with an atlas $\{f_\ell, V_\ell\}$, where $\{V_\ell\}$ is an open cover of M and each f_ℓ is a diffeomorphism form V_ℓ to a bounded open set $W_\ell \subset \mathbb{R}^d$. Functions on $\mathbb{R}^d$ can be pulled back via $f_\ell^* : C^\infty(W_\ell) \to C^\infty(V_\ell)$. The canonical lift of f_ℓ between T^*V_ℓ and T^*W_ℓ is denoted by $\tilde{f}_\ell$:

$$(x,\xi) \in T^*V_\ell \mapsto \tilde{f}_\ell(x,\xi) = (f_\ell(x), (Df_\ell(x)^{-1})^T\xi) \in T^*W_\ell\,,$$

where A^T denotes the transpose of A. Its corresponding pull-back will be denoted by $\tilde{f}_\ell^* : C^\infty(T^*W_\ell) \to C^\infty(T^*V_\ell)$. A smooth partition of unity adapted to the cover $\{V_\ell\}$ is a set of functions $\phi_\ell \in C_c^\infty(V_\ell)$ such that $\sum_\ell \phi_\ell = 1$ on M.

Any observable (i.e. a function $a \in C^\infty(T^*M)$) can now be split into $a = \sum_\ell a_\ell$ where $a_\ell = \phi_\ell a$, and each term pushed to $\tilde{a}_\ell = (\tilde{f}_\ell^{-1})^* a_\ell \in C^\infty(T^*W_\ell)$. If a belongs to a standard class of symbols, for instance

$$a \in S^{m,k} = S^k(\langle\xi\rangle^m) \stackrel{\text{def}}{=} \left\{ a = a_\hbar \in C^\infty(M),\ |\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha,\beta}\hbar^{-k}\langle\xi\rangle^{m-|\beta|} \right\},$$

each a_ℓ can be be Weyl-quantized into a pseudodifferential operator on $\mathcal{S}(\mathbb{R})$ via the formula

$$\forall u \in \mathcal{S}(\mathbb{R}^d),\ \mathrm{Op}_\hbar^w(\tilde{a}_\ell)u(x) = \frac{1}{(2\pi\hbar)^d}\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle x-y,\xi\rangle}\,\tilde{a}_\ell\left(\frac{x+y}{2},\xi;\hbar\right)u(y)\,dy\,d\xi$$

To pull-back this operator on $C^\infty(V_\ell)$, one first takes another smooth cutoff $\psi_\ell \in C_c^\infty(V_\ell)$ such that $\psi = 1$ in a neighbourhood of $\operatorname{supp}\phi_\ell$. The quantization of $a \in S^{m,k}$ is finally defined by gluing local quantizations together, yielding to

$$\forall u \in C^\infty(M), \mathrm{Op}_\hbar(a)u = \sum_\ell \psi_\ell \times f_\ell^* \circ \mathrm{Op}_\hbar^w(\tilde{a}_\ell) \circ (f_\ell^{-1})^*(\psi_\ell u)$$

The space of pseudodifferential operators obtained from $S^{k,m}$ by this quantization will be denoted by $\Psi^{m,k}$. Although this quantization depends on the cutoffs, the principal symbol map $\sigma : \Psi^{m,k} \to S^{m,k}/S^{m,,k-1}$ is intrinsically defined and do not depend on the choice of coordinates. The residual class is made of operators in the space $\Psi^{m,-\infty}$. As an example, the (semiclassical) Laplacian $-\hbar^2\Delta_g \in \Psi^{0,2}$ is a pseudodifferential operator, and its principal symbol is given by $\sigma(-\hbar^2\Delta_g) = \|\xi\|_g^2 = g_x(\xi,\xi) \in S^{2,0}$.

In this article, we are concerned with a purely semiclassical theory and then deal only with compact subsets of T^*M. If $A \in \Psi^{m,k}$, we will denote by $\mathrm{WF}_\hbar(A)$ the semiclassical wave front set of A. A point $\rho \in T^*M$ belongs to $\mathrm{WF}_\hbar(A)$ if for some choice of local coordinates near the projection of ρ, the full symbol of A is in the class $S^{m,-\infty}$. $\mathrm{WF}_\hbar(A)$ is a closed subset of T^*M, and $\mathrm{WF}_\hbar(AB) \subset \mathrm{WF}_\hbar(A) \cap \mathrm{WF}_\hbar(B)$. In particular, if $\mathrm{WF}_\hbar(A) = \emptyset$, then A is a negligible operator, i.e. $A \in \Psi^{m,-\infty}$. If $\Psi \in L^2(M)$, we also define the semiclassical wave front set of Ψ by :

$$\mathrm{WF}_\hbar(\Psi) = \left\{ (x,\xi)\ :\ \exists a \in S^{m,0},\ a(x,\xi) \neq 0,\ \|\operatorname{Op}_\hbar(a)\Psi\|_{L^2(M)} = \mathcal{O}(\hbar^\infty) \right\}^{\mathbf{c}}$$

where the superscript $^{\mathbf{c}}$ indicates the complementary set. We will often make use of the following fundamental propagation property : if U^t is a Fourier integral operator associated to a symplectic diffeomorphism $\Phi^t : T^*M \to T^*M$, then

$$\mathrm{WF}_\hbar(U^t\Psi) = \Phi^t(\mathrm{WF}_\hbar(\Psi)).$$

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Institut de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette, France
E-mail address: emmanuel.schenck@cea.fr